Periodic Space-Time Formulation for Numerical AC Loss Computation in Superconductors

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***Abstract*—In this paper we present a new model for computing the current density and field distributions in superconductors by means of a finite element (FE) periodic space-time (PST) formulation. By considering the time as a space dimension, and considering periodic excitations for the applied field or transport current, we can use a static model to solve this time dependent problem. This approach has the potential to overcome one of the major problems of FE modeling of superconductors: the length of simulations, even for relatively simple cases. In this paper we show our first results for different cases of superconductors with AC magnetic fields and currents. Results are compared with those of standard time-dependent methods and analytical solutions.**

***Index Terms*—AC losses, finite-element method, numerical methods, space-time.**

I. Introduction

FINITE-ELEMENT calculations have proved to be capable of accurately predicting the AC losses in high-temperature superconductor (HTS) devices of increasing complexity. Different formulations (using different state variables) are possible, but in general the FE models solve Faraday's law (which contains partial derivatives with respect to position and time), using a non-linear resistivity for the superconductor material: $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, where $\mathbf{E} = \rho \mathbf{J}$. The resistivity ρ is described by a non-linear function of the current density: $\rho(J) = E_c/J_c|J/J_c|n - 1$.

The presence of a non-linear resistivity does not allow for a steady-state time harmonic solution, so the standard approach is to solve it as a time-dependent problem. This means that several time-steps have to be solved: typically two cycles are solved to eliminate the effects of the transient from the initial condition (generally a material in its virgin state). This approach results in a lengthy solving process. Even simple problems like a rectangular wire carrying transport current and/or subjected to an external field require several minutes of computation on fast workstations. The simulation of real wires and devices would involve a large increase in the number of mesh nodes, due either to the geometry of the wire itself (the very thin YBCO coated conductors are the most promising wires) or to the high number of tapes in complex devices (cables, coils). For the design of practical devices, numerous configurations and parameters typically have to be tested, so the standard approach based on a solution of the transient problem, while providing accurate results, is not optimal, due to the unreasonable size of the computational times. Therefore the search for an alternative, faster and equally reliable way of computing current and field profiles as well as AC losses in superconducting devices is mandatory.

In this framework, we started developing, implementing and testing a new model based on a periodic space-time (PST) formulation, where one of the dimensions of the simulated domain represents the time [1], [2]. This allows one to solve the problem as a static one, which can be better handled by the solvers and is likely to be faster.

In this paper we present the first results of this project, showing that the problem is well posed and that there is a potential for obtaining a faster simulation tool. The model has been implemented in the commercial finite-element software package COMSOL Multiphysics [3].

II. Model Description and Results

For our initial analysis we considered two problems, an infinite slab subjected to a parallel external AC field and a round wire carrying a transport current. The first is an example of a 1-D transient problem that is transformed into a 2-D static one. The second is a 2-D transient problem (the circular cross-section wire), transformed into a 3-D static one (a cylinder, whose axis defines the time). The common point is that the boundary conditions for the magnetic field are known in both cases, so they serve as an initial test of the correctness of the implementation of the model. In fact, one of the most important contributions of this paper is the extension of the concept of the PST formulation to the case of multiple conductors of rectangular shape carrying an arbitrary current. In that case, the boundary conditions for the magnetic field are unknown, and the currents have to be imposed by means of integral constraints.

A. HTS Slab in AC Magnetic Field

The problem of an infinite slab subjected to an external AC magnetic field is one-dimensional and Faraday's equation reduces to the scalar diffusion equation:

$$-\frac{\partial}{\partial x}\left[\frac{\rho}{\mu_0}\frac{\partial B}{\partial x}\right] + \frac{\partial B}{\partial t} = 0, \tag{1}$$

where the quantity $\rho/\mu_0 = f(J, B) = f(\partial B/\partial x, B)$ is known as the diffusion coefficient (highly nonlinear in this case).

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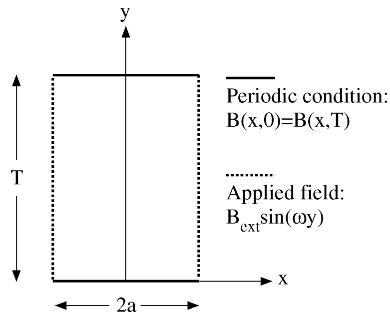

Fig. 1. Schematic view of the geometry and boundary conditions used for the PST model of an infinite slab subjected to a parallel AC field.

Fig. 1 schematically shows the geometry used for the PST model. The width of the slab $(2a)$ is represented on the x axis, whereas the time corresponds to the y axis. As for the boundary conditions, we know that the field is equal to the applied one at the slab's edges at each time instant, so we applied the Dirichlet condition $B = B_{ext} sin(\omega y)$ on the vertical sides of the domain. Since one cycle of the field oscillation is simulated, the solution at $y = 0$ must be the same at $y = T$, where T is the period of the cycle. This is done by imposing a periodicity condition on the two horizontal sides of the domain: $B(x, y = 0) = B(x, y = T)$. The problem size can be further reduced by considering only a half-period and imposing a condition of anti-symmetry: $B(x, y = 0) = -B(x, y = T/2)$. This allows simulating a smaller domain, using less mesh nodes and, consequently, reduce by half the computing time. It has to be remarked that in general the x and y scales might differ by several orders of magnitude, depending on the considered dimensions of the slab and frequency of the applied field. Since the aspect ratio of the mesh elements must be close to one to avoid badly conditioned matrices in the numerical problem, it is important that both x and y be on the same scale. Failure to do so will result in a very elongated rectangle with badly shaped mesh elements; if one wants to use a similar mesh density on both sides the number of mesh nodes is very large. In order to avoid these problems, we scaled the time by a factor c, so that the simulated domain is square and the mesh elements are regular.

Fig. 2 shows the computed magnetic flux density (a) and current density (b) distributions in a superconducting slab for an applied field of 40 mT at 500 Hz. The slab width is 1 mm, the superconductor's parameter are $J_c = 10^8 \text{ A/m}^2$, $n = 25$. In Fig. 2(b) the current-free region can be noticed in the center, whereas on the sides the current density changes sign as a function of the time (vertical direction).

We have compared the results of the current/field profiles and AC losses obtained with the PST model with those obtained with a standard transient model [4] and with analytical models [5]. As an example we show the AC losses. Generally the AC losses are computed by integrating the instantaneous power dissipation ρJ^2 on the conductor's cross section and then over a cycle. In order to obtain the value in watt per cubic meter, the integral has to be multiplied by the frequency and by the conductor's cross section (the slab width in this case). With the PST model this is done by integrating the quantity $2f\rho J^2/(2ac)$ on the simulated domain: the factor 2 takes into account that we simulate half a cycle, whereas the factor c takes into account the fact that on the y axis the "real" time has been scaled $(y = ct)$. The results are summarized in Fig. 3, where the losses are also compared with those computed by analytical expression [5]. Computing times refer to a workstation equipped with a 2.4 GHz processor and 3 GB of RAM. It can be noticed that the agreement on the loss value between the different models is very good. As for the computing times, a remark is necessary. Time-dependent models compute the solution of the problems on several cycles (typically two, to avoid transient effects), starting with an initial condition where all the variables are set to zero. This means that when a parameter (e.g. the amplitude of the applied field) is changed, the solution has to be computed from zero initial conditions every time. This is not optimal in view of using the model for design and optimization purposes, where typically many configurations and working conditions have to be tested. On the other hand, the PST formulation allows computing a solution starting from a previous one (as it uses a static solver), where a parameter is only slightly different. Since the simulation starts from a value quite close to the solution one is looking for, the computing time is lower. For the PST formulation, Fig. 3 reports the computing times for the simulations performed starting

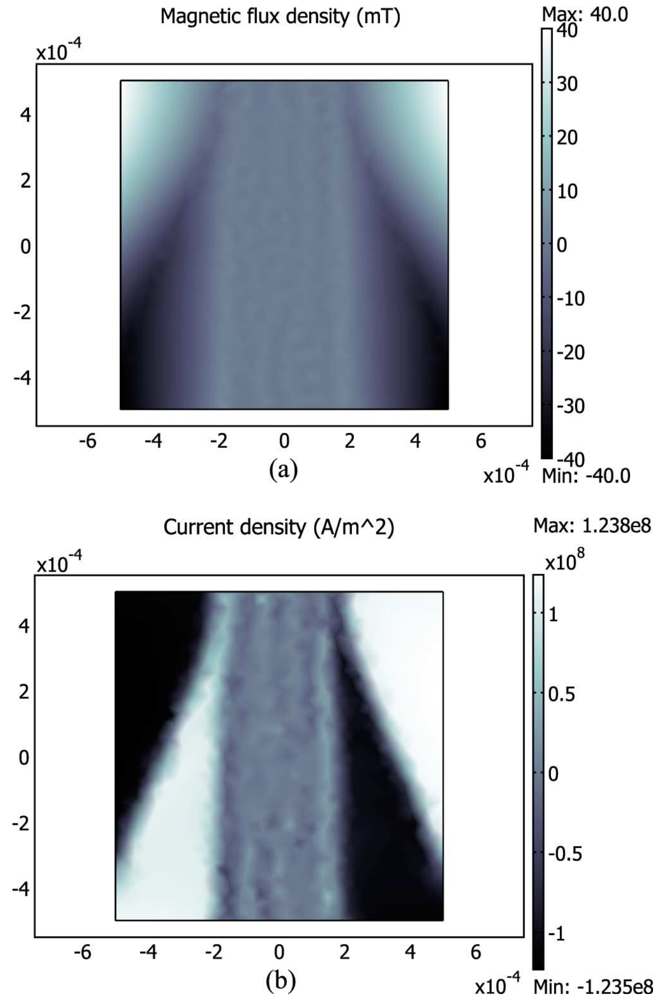

Fig. 2. Magnetic flux density (a) and current density (b) distribution in a superconducting slab for an applied field of 40 mT at 500 Hz. The vertical direction represents the time (in seconds).

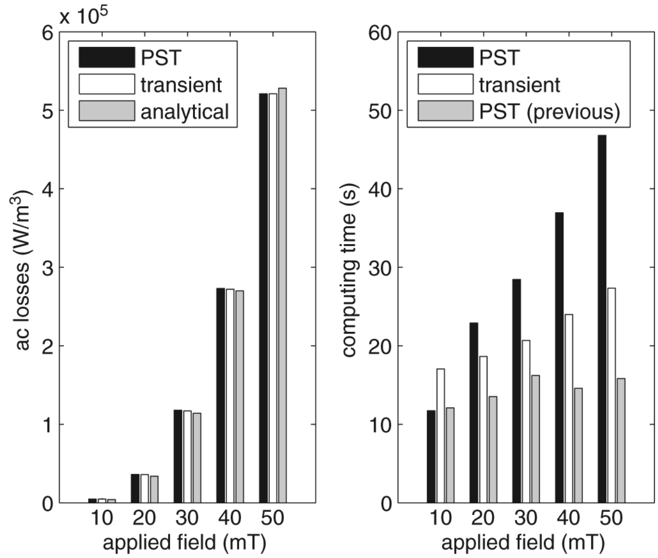

Fig. 3. Comparison of AC losses (in $\mathrm{W/m^3}$) and computing times obtained with different models for a superconducting slab subjected to an external AC magnetic field.

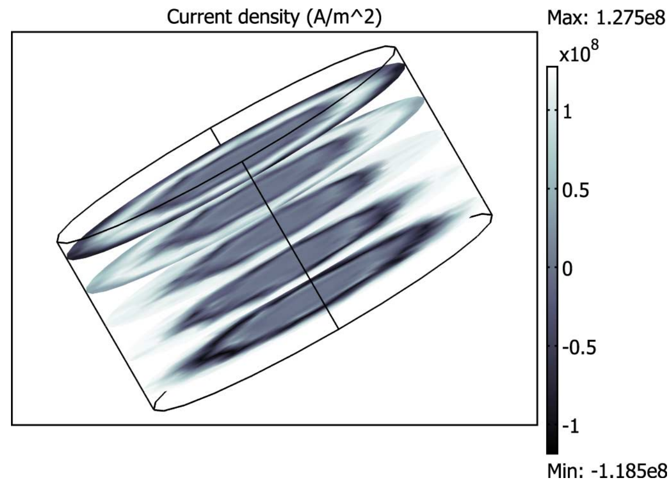

Fig. 4. Current density distribution in a round conductor at five different instants of the half-period obtained with the PST.

from zero and from a previous solution (i.e. a previous value of the field). It can be seen that in the first case the computing time increases with the field value, whereas in the second case it is quite constant. This is a relatively simple problem, so the gain in computing time might not seem important. However, it is expected to be more relevant in practice where more complex problems will need to be considered.

B. Round Conductor Carrying AC Current

In order to test the PST approach on a higher dimensional problem (2-D geometry + time), we implemented the model for the case of a round conductor carrying AC current. Even if there is one more dimension to handle, this problem is conceptually similar to the case of the slab, because we know the boundary conditions for the state variable. For a given transport current of amplitude I_a, the magnetic field on the conductor's boundary is tangential and has amplitude $H_\Phi = I_a/(2\pi r)$, where r is the radius of the round conductor. An example of the current density distribution at different five instants of the half-period is given in Fig. 4. The conductor has a radius of 1 mm, $J_c = 10^8\ \mathrm{A/m^2}$, $n = 25$, and carries a (peak) transport current of 200 A. The critical current is 314 A. Once again, we obtained good agreement for the AC loss computation with all methods (except at current close to I_c, where Norris's formula diverges). The reduction of computing time starting from the previously computed solution is even more evident than for the slab case-see Fig. 5.

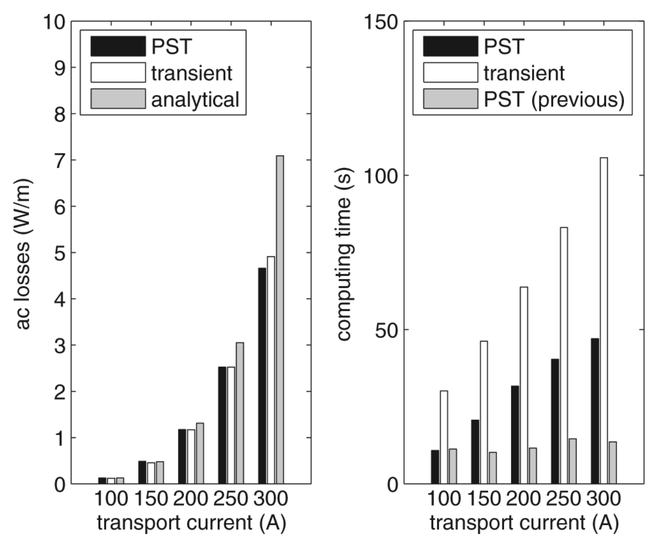

Fig. 5. Comparison of AC losses (in W/m) and computing times obtained with different models for a round conductor carrying AC current. Analytical data are computed by Norris's formula for an elliptical wire [6].

C. Multiple Conductors of Arbitrary Shape

The tests described above assumed the knowledge of the magnetic field amplitude at the conductor's edges. In the presence of multiple conductors this is not the case. Even the simple case of only one conductor with rectangular cross-section cannot be simulated with the above implementation, because the magnetic field at the edges is unknown (demagnetizing factor). The basic idea for generalizing the model is to simulate the conductor(s) and the insulating domain around them and to impose the desired transport current by means of time-dependent constraints, and leaving the field at the external boundary free to vary by imposing a Neumann boundary condition. These constraints are to be applied to the individual conductors, so that we can simulate the most general case of multiple conductors carrying an arbitrary current. This will be useful to simulate for example the cross-section of a multiple-turn pancake coil, where each conductor (i.e. each turn of tape) carries the same current and experiences the magnetic field generated by the neighbors or multi-layer cables, where the tapes of each layer carry in general different currents.

As a first and relatively simple approach to this problem, we chose to use a finite difference technique to deal with the time derivative term, which leads to the discretization of the time transient problem into a finite number of "slices" or "layers" along the time axis.

We start from the general diffusion equation:

$$\nabla \times \frac{\rho}{\mu_0} \nabla \times \mathbf{B} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2}$$

with the Neumann boundary conditions

$$\frac{\rho}{\mu_0} \frac{\partial \mathbf{B}}{\partial \mathbf{n}} = \mathbf{0}, \tag{3}$$

where $\mathbf{n}$ is the unit vector normal to the external face of the domain. We then approximate the variation rate of the magnetic flux with a classical second order finite difference formula, i.e.

$$\frac{\partial \mathbf{B}}{\partial t} \approx \frac{\mathbf{B}(t+\Delta t) - \mathbf{B}(t-\Delta t)}{2\Delta t}. \tag{4}$$

For each layer j we have a solution $\mathbf{B}_j = \mathbf{B}(x, y, t_j)$, corresponding to an equation of the form:

$$\nabla \times \frac{\rho}{\mu_0} \nabla \times \mathbf{B}_j = \frac{\mathbf{B}_{j+1} - \mathbf{B}_{j-1}}{2\Delta t}. \tag{5}$$

Using anti-symmetric boundary conditions with n_t points per half-cycle, this allows setting-up a problem involving n_t "layers" coupled together by the $\mathbf{B}_{j\pm1}$ terms appearing on the right hand-side, i.e.

$$\begin{aligned} \nabla \times \frac{\rho}{\mu_0} \nabla \times \mathbf{B}_1 &= (\mathbf{B}_2 - \mathbf{B}_0)/(2\Delta t), \\ \nabla \times \frac{\rho}{\mu_0} \nabla \times \mathbf{B}_2 &= (\mathbf{B}_3 - \mathbf{B}_1)/(2\Delta t), \\ &\cdots \\ \nabla \times \frac{\rho}{\mu_0} \nabla \times \mathbf{B}_{n_t} &= (\mathbf{B}_{n_t+1} - \mathbf{B}_{n_t-1})/(2\Delta t). \end{aligned} \tag{6}$$

with $\mathbf{B}_0 = -\mathbf{B}_{n_t}$ and $\mathbf{B}_{n_{t+1}} = -\mathbf{B}_1$ because of antisymetry.

In COMSOL, this approach can be implemented by using extrusion coupling variables to make the variables $\mathbf{B}_{j+1}$, $\mathbf{B}_{j-1}$ available in the jth "layer" (or "geometry" in COMSOL's terminology) and by inserting the finite difference expressions as source terms in a COMSOL's magnetostatic application mode. This means that instead of having a full 3-D geometry, we have a set of n_t 2-D coupled problems to solve.

As an example, we considered three stacked straight conductors (side dimension 1 mm, distance center-to-center 2 mm) carrying sinusoidal currents of different amplitudes: 80, 70, and 60 A. We compared the results obtained with this finite difference implementation of the PST model with those obtained by a standard transient 2-D method, and found good agreement. As an example, Fig. 6 shows the instantaneous losses in the three conductors during a half-cycle. The frequency of the sources is 100 Hz, the critical current of each conductor is 100 A. The time domain has been "sliced" in 16 time steps, whereas for sake of accuracy in the transient model we have used 100 steps per cycle.

This therefore shows that the PST method is applicable in the general case, but the finite difference approach is not enough

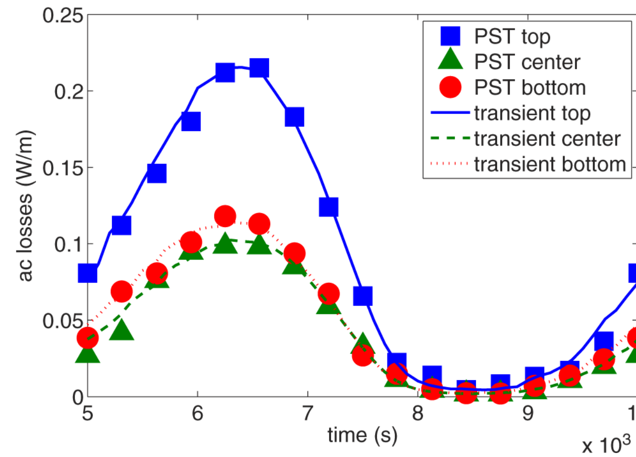

Fig. 6. Instantaneous AC losses for three stacked superconductors carrying current of 80, 70, and 60 A computed with the PST formulation (symbols) and the standard transient model (lines).

flexible to provide the expected benefits in terms of reduction of computation times (see comments below about further work).

III. Discussion and Conclusion

In this paper we presented the idea of using a periodic space-time (PST) formulation to compute electromagnetic quantities in HTS for power applications. The preliminary results shown here were intended to demonstrate the correctness of this new approach. Further work is needed to optimize the most general case of multiple conductors carrying arbitrary currents, which is also the most interesting for practical applications such as cables and coils. The ideal solution would be to consider a full 3-D geometry and mesh and impose the current constraints in each conductor by means of weak form constraints. This would allow beneficiating from powerful mesh adoption tools developed for static solvers, which are expected to lead to significant reduction of computation times.

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