



AN ADAPTIVE VERSION OF GLIMM'S SCHEME*

Dedicated to Professor James Glimm on the occasion of his 75th birthday

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Abstract This article describes a local error estimator for Glimm's scheme for hyperbolic systems of conservation laws and uses it to replace the usual random choice in Glimm's scheme by an optimal choice. As a by-product of the local error estimator, the procedure provides a global error estimator that is shown numerically to be a very accurate estimate of the error in $L^1(\mathbb{R})$ for all times. Although there is partial mathematical evidence for the error estimator proposed, at this stage the error estimator must be considered ad-hoc. Nonetheless, the error estimator is simple to compute, relatively inexpensive, without adjustable parameters and at least as accurate as other existing error estimators. Numerical experiments in 1-D for Burgers' equation and for Euler's system are performed to measure the asymptotic accuracy of the resulting scheme and of the error estimator.

Key words

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1 Introduction

Consider an approximate solution w generated by Glimm's scheme for the entropy solution $v : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ of a system of n nonlinear conservation laws

$$v_t + f(v)_x = 0, \quad (1)$$

$$v(\cdot, 0) = v_0(\cdot) \quad (2)$$

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where f is strictly hyperbolic with either genuinely nonlinear or linearly degenerate fields and initial data v_0 belonging to $BV(\mathbb{R}) \cap L^1(\mathbb{R})$. We aim to construct local and global a posteriori error estimators in $L^1(\mathbb{R})$ for Glimm's scheme and use these to transform the scheme into a deterministic and more reliable scheme.

Quoting Smoller [37], "it is fair to say that all of the subsequent work on systems of hyperbolic conservation laws is based upon" Glimm's "profound paper" of 1965 [15]. Glimm's unconventional scheme has unique properties that make it useful for the qualitative analysis of entropy solutions to conservation laws but as a numerical tool, it is far from ideal. Similar in form to Godunov's scheme, it replaces the conservation principle for the selection of the new states by a random choice among the states in the entropy solution of the Riemann problem at each cell interface. Improving on Glimm's proof that his scheme converged for all but a set of measure zero of random sequences, Liu introduced his wave-tracing technique to show that it was sufficient to pick an equidistributed sequence [33]. Chorin [8] and Collela [10] performed numerical experiments with Glimm's scheme and argued that, with some minor modifications, the scheme could be used as a numerical scheme because of its sharp resolution of discontinuities. The front-tracking method, developed by several researchers [5, 13, 14, 36], is somewhat simpler than Glimm's scheme and yet possesses most of the same useful properties. From the numerical point of view, the front-tracking scheme is a much more efficient tool although its numerical implementation is rather awkward [19] and its order of convergence is identical to Glimm's theoretical order of $O((1/\sqrt{\Delta x}) \ln \Delta x)$. A large literature exists for order of convergence estimates for Glimm's scheme, of which we mention only [18, 35, 38] and the optimal results of Bressan and Marson [7] in the case of systems. Nonetheless, it is still the only scheme for which convergence has been demonstrated for special classes of conservation laws, such as nonconvex ones [30], and the only one for which there is a stability theory for nonlinear systems of conservation laws [11, 21, 34]. It is therefore still pertinent to attempt to understand and improve the accuracy of Glimm's scheme.

Building upon the success of a posteriori error estimation techniques for elliptic equations [2], several researchers attempted to construct error estimates for numerical schemes for nonlinear hyperbolic differential equations, but with varying levels of success. Those that approached the problem using adjoint-based error estimators obtained quite accurate but costly estimators for weakly nonlinear problems, that is where linearization around the entropy solution was justified [4, 20, 23]. These estimators accounted for error propagation and cancellation in the estimation of functionals of the solution but involved the weaker stability theory of Friedrichs in negative index Sobolev spaces which were unnatural for conservation laws. This approach for nonlinear conservation laws with shocks is so far mathematically justified only for scalar equations in 1D [20, 38]. Those that approached the problem using Kruskov's stability theory in L^1 [9, 16, 24] obtained formal a posteriori error estimators for scalar conservation laws in several space dimensions but the estimators developed in this way are not very accurate and do not account for error cancellation. Others argued that certain schemes, such as discontinuous Galerkin methods, possessed superconvergence properties [1] that allowed one to estimate the error locally. Unfortunately, this third approach still relies on unique properties of the numerical scheme and only holds for very smooth and uniform meshes. Finally, one of the authors has attempted to use stability theories specifically designed for nonlinear systems of conservation

laws [25, 26, 27] in order to obtain qualitative error estimates that were valid in the nonlinear regime. Unfortunately, the rigorous error estimators for systems [25] overestimated the error because they failed to account for error cancellation.

In this paper, we propose an ad-hoc adaptation of the error estimator in [26] for Glimm's scheme. In contrast to the work [26] where Glimm's scheme was described with the help of Liu's wave-racing method [32], Glimm's scheme is formulated here as a standard finite difference scheme. Roughly speaking, Glimm's scheme is a staggered finite difference scheme where the values of the solution at the next timestep are sampled randomly at the position $\theta\Delta x \in [-\Delta x, \Delta x]$ inside the entropy solution of the Riemann Problem, see Section 2. In this paper, we construct a local error estimator at each cell interface containing a Riemann problem and choose the optimal θ so that the local error is minimized. The accuracy of the scheme comes from the fact that errors are propagated and can either accumulate or cancel, depending on the wave interactions present in the solution. The final result is a robust version of Glimm's scheme which comes with the added benefit of having an error estimator. It is also noted that the error estimator has no adjustable parameters (in contrast to [4, 16, 23, 24]). The proposed scheme also has many similarities with the reservoir technique [3] where local "CFL" counters are introduced to implement wave-dependent timestepping.

As of this writing, the error estimator has a rigorous mathematical foundation only in the case of approximate solutions generated by Glimm's scheme and described by Liu's wave tracing method. Moreover, the mathematical result so far holds only for scalar conservation laws with initial data containing only shocks [26]. Numerical experiments reported in [27] indicated that the theory should continue to hold even for arbitrary initial data and this is the subject of ongoing work by Laforest and LeFloch. We emphasize that the earlier use of Liu's wave tracing method in the numerical results [27] required an initial decomposition of the all of the jumps in the initial data into a set of consitutive waves and the knowledge of all the random choices a priori. In this sense, the error estimation performed with the help of Liu's wave tracing method required a computational burden similar to the cost of determining the domain of dependence of each wave, as in [24], or the computation of an adjoint problem [20]. In the construction described here, the error estimator is computed a posteriori at each timestep using appropriately computed sum of previous errors but without knowledge of the past or future of the approximate solution. This property is the main novelty of the proposed scheme.

The paper is organized as follows. In Section 2, the authors describe Glimm's scheme and review the relevant error estimators they have already constructed for conservation laws. In Section 3, the new algorithm is described in detail. In Section 4, the authors present numerical results for Burgers' equation and Euler's system of equations for an ideal gas which support the claim of accuracy of their error estimator.

2 Preliminaries

2 Riemann problems

We begin by reviewing the theory of conservation laws to set the notation and the assumptions necessary for this work.

Consider a system of n coupled nonlinear conservation laws (1) where the Jacobian $Df(u)$ of the smooth function f has n distinct real eigenvalues $\lambda^{(1)}(u) < \dots < \lambda^{(n)}(u)$ for u inside some neighborhood of the origin $\Omega \subset \mathbb{R}^n$. Given initial data $v_0 : \mathbb{R} \rightarrow \Omega$ we shall say that the bounded measurable function $v : \mathbb{R}^+ \times \mathbb{R} \rightarrow \Omega$ is a *weak solution* of the initial value problem (1) if

$$\int_0^\infty \int_{-\infty}^\infty (\phi_t v + \phi_x f(v)) \, dx dt + \int_{-\infty}^\infty \phi(x, 0) v_0(x) \, dx = 0 \quad (3)$$

for every smooth ϕ with compact support in $t \geq 0$. Among the class of solutions in $v \in L^\infty([0, T], BV(\mathbb{R})) \cap \text{Lip}([0, T], L^1(\mathbb{R}))$ with sufficiently small total variation and for each pair (ν, η) satisfying $\eta'(u) = Df(u)\nu'(u)$ for $u \in \Omega$, there exists a *unique* and *stable* weak solution of (1) that satisfies, in a weak sense, the inequality

$$\nu(v)_t + \eta(v)_x \leq 0.$$

This solution v is called the entropy solution.

Let $r_1, \dots, r_n$ be the eigenvectors of $Df(u)$, given as smooth functions of $u \in \Omega$. Define the k -th rarefaction curve through the point u^- to be the unique solution $R_k(\cdot)(u^-)$ of

$$\frac{d}{d\sigma} R_k(\sigma)(u^-) = r_k(R_k(\sigma)(u^-)), \quad \text{and} \quad R_k(0)(u^-) = u^-.$$

Given two states u^+ and u^- , the matrix

$$A(u^+, u^-) = \int_0^1 Df((1-\theta)u^- + \theta u^+) \, d\theta \quad (4)$$

satisfies the relation

$$A(u^+, u^-)(u^+ - u^-) = f(u^+) - f(u^-). \quad (5)$$

Define the k -th shock curve through u^- to be the set of states $u^+ \equiv S_k(\sigma)(u^-)$ satisfying the Rankine-Hugoniot condition

$$f(u^+) - f(u^-) = s(u^+ - u^-),$$

and such that $s \in \mathbb{R}$ is the k -th eigenvalue of $A(u^+, u^-)$. For the k -th shock curve, the scalar s will be called the Rankine-Hugoniot shock speed and denoted $s^{(k)}(u^+, u^-)$.

We say that a family $k \in \{1, \dots, n\}$ is *genuinely nonlinear* if $r_k \cdot \nabla \lambda^{(k)} \neq 0$, and linearly degenerate if $r_k \cdot \nabla \lambda^{(k)} = 0$. If the k -th family is genuinely nonlinear then the shock and rarefaction curves can be parameterized to satisfy

$$\begin{aligned} \frac{d}{d\sigma} \lambda^{(k)}(S_k(\sigma)(u)) &= 1, & \frac{d}{d\sigma} \lambda^{(k)}(R_k(\sigma)(u)) &= 1, \\ \lambda^{(k)}(S_k(\sigma)(u)) - \lambda^{(k)}(u) &= \sigma, & \lambda^{(k)}(R_k(\sigma)(u)) - \lambda^{(k)}(u) &= \sigma. \end{aligned}$$

For linearly degenerate families, parameterization can be arc-length but other more natural choices may exist for specific system, like density for the 2-nd wave family in Euler's system. For a genuinely nonlinear family k define

$$T_k(\sigma)(u^-) = \begin{cases} R_k(\sigma)(u^-), & \sigma \geq 0, \\ S_k(\sigma)(u^-), & \sigma < 0, \end{cases} \quad (6)$$

and for a linearly degenerate family let $T_k(\sigma)(u^-) = S_k(\sigma)(u^-)$. The curve T_k is smooth for $\sigma \neq 0$ with two continuous derivatives at $\sigma = 0$ [6], a fact that will be useful later.

A Riemann problem is an initial value problem for (1) consisting of piecewise constant initial data along the $t = 0$ axis formed of two constant states u^- and u^+ separated at the origin. It is well-known [29] that if u^- and u^+ belong to a sufficiently small neighborhood of the origin in $\mathbb{R}^n$ then the Riemann problem has a unique self-similar solution composed of $n + 1$ constant states $u_0 = u^-, u_1, \dots, u_n = u^+$ satisfying $u_k = T_k(p_k)(u_{k-1})$ for real numbers $p_k = p_k(u^-, u^+)$. The solution is formed of n self-similar regions where it takes on the values u_{k-1}, u_k along the boundaries of the k -th region. Each region contains either one

- discontinuity traveling with the Rankine-Hugoniot speed and separating u_{k-1} from

$$u_k = S_k(p_k(u^-, u^+))(u_{k-1});$$

- or a continuous solution $u(x/t)$ satisfying $\lambda^{(k)}(u(x/t)) = x/t$ and

$$u_k = R_k(p_k(u^-, u^+))(u_{k-1}).$$

For more information on hyperbolic conservation laws one may consult [6].

2.2 Glimm's scheme

A detailed description of Glimm's scheme [15] can be found in many references such as Smoller's treatise [37] or Bressan's recent book [6]. Below, we give only the barest of information and refer the reader to other sources for more information.

Given initial data $v_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, we begin the approximation by choosing an initial discretization of the entire space $\mathbb{R}$ into intervals $I_i := [x_{i-1}, x_{i+1})$ where i is an odd integer and $x_i := i\Delta x$ for some fixed Δx . The temporal discretization Δt is then chosen to satisfy the so-called Courant–Friedrichs–Levy (CFL) condition

$$\sup_{u,k} |\lambda^{(k)}(u)| \leq \frac{\Delta x}{\Delta t}, \quad (7)$$

where the supremum is taken over $u \in \Omega$ and $k \in \{1, \dots, n\}$. Finally, we construct the initial data for Glimm's scheme $w(\cdot, 0)$ as a piecewise constant approximation to v_0 that is constant along the intervals I_i with i odd.

In Glimm's scheme, we assume that the approximate solution w is known at time $t_j := j\Delta t$ and is piecewise constant over the sequence of intervals I_{i-1} with $i+j$ even. We then use $w(\cdot, t_j)$ as initial data for (1) along the line $t = t_j$ and let w be the entropy solution inside the strip $\mathbb{R} \times J_j$ where $J_j := [t_j, t_{j+1})$. The CFL condition (7) implies that the solution inside the strip $\mathbb{R} \times J_j$ is simply the solution of a sequence of independent Riemann problems originating from the nodes (x_i, t_j) . This solution is already known to exist if the total variation of the initial data is sufficiently small. In order to make $w(\cdot, t_{j+1})$ into a piecewise constant function on the intervals I_i along the line $t = t_{j+1}$, Glimm suggested picking a random number $\theta_{i,j+1} \in [-1, 1]$ and defining

$$w(x, t_{j+1}) = \lim_{s \rightarrow t_{j+1}-} w(x_i + \theta_{i,j+1}\Delta x, s), \quad \forall x \in I_i. \quad (8)$$

This new solution at time t_{j+1} is then piecewise constant over a sequence of intervals staggered with respect to those at time t_j . A priori bounds can be found that show that, if the total

variation is small, then this procedure can be repeated indefinitely to define a piecewise constant solution at all later times. In this paper, we will show that the random choice can be replaced by a choice that is in some way locally optimal.

In terms of numerics, the implementation of Glimm's scheme requires a complete description of the entropy solution of the Riemann problem at each cell interface, a task which can be costly in practice. On the other hand, Harten and Lax have shown that Glimm's scheme still converges if the exact Riemann solver is replaced by an approximate Riemann solver [17].

2.3 Residuals and discrepancies

In this section we present our local error estimators and relate them to error estimators introduced by others. Roughly speaking, this work uses the classical residual of an approximation w , which for conservation laws is

$$R(w) := w_t + f(w)_x. \quad (9)$$

This work requires a somewhat nuanced interpretation of this quantity for two reasons. First of all, because this quantity must be interpreted in a distributional sense and secondly because the integral of this quantity could vanish when unphysical rarefaction shocks replace continuous rarefaction waves. The second difficulty originates from the fact that (9) only measures conservation and not entropy production. Despite these observations, the residual possesses additivity properties which make it a more practical error estimator and entropy production can be measured indirectly [25], as we explain below.

We begin by analyzing the residual (9) in the context of scalar conservation laws, as was done in [26]. Afterwards, a natural extension to systems will be described and contrasted to the error estimator in [25].

Lemma 2.1 [25] In a neighborhood of the line $t = t_j$, the residual of an approximate solution w obtained by Glimm's scheme is

$$R(w)(x, t) := \delta_{t_j}(t) \left(\lim_{s \rightarrow t_j^-} w(x, s) - w(x, t_j) \right). \quad (10)$$

Lemma 2.2 [25, 26] Consider a strictly convex scalar conservation law (1), i.e. (1) with $n = 1$ and $f'' > 0$. Assume that w is an approximate solution obtained by Glimm's scheme and that, in the notation of Section 2.2, w^- and w^+ are respectively the left and right hand states of the Riemann problem at (x_i, t_j) , $i + j = 1 \bmod 2$.

1) When the solution is a shock then

$$\iint_{I_i \times J_{j+1/2}} R(w)(x, t) \, dx dt = (w^+ - w^-) (\Delta x \operatorname{sign}(s \Delta t - \theta_{i,j+1} \Delta x) - s \Delta t) \quad (11)$$

where $s = (f(w^+) - f(w^-)) / (w^+ - w^-)$.

2) When the solution is a rarefaction

$$\iint_{I_i \times J_{j+1/2}} R(w)(x, t) \, dx dt = \int_{-\Delta x}^{\Delta x} V(x) - V(\theta_{i,j+1} \Delta x) \, dx \quad (12)$$

where

$$V(x) = \begin{cases} w^- & \text{if } x < x_i + f'(w^-) \Delta t, \\ (f')^{-1} \left(\frac{x - x_i}{t - t_j} \right) & \text{if } x \in [x_i + f'(w^-) \Delta t, x_i + f'(w^+) \Delta t), \\ w^+ & \text{if } x_i + f'(w^+) \Delta t \leq x. \end{cases}$$

This theorem shows that despite the fact that the error is distributional, it's integral in space and time is a well-defined quantity. For rarefaction waves in the scalar case, it is clear that if $\theta_{i,j+1}\Delta x$ is chosen carefully, then the integral of the residual will vanish. In [25], this issue was avoided by using the following larger quantity as an error estimator for rarefactions:

$$\int_{-\Delta x}^{\Delta x} |V(x) - V(\theta_{i,j+1}\Delta x)| \, dx. \quad (13)$$

In unpublished work of Laforest, this quantity was shown to be a measure of entropy production, as defined by Dafermos [12], with respect to the family of Kruskov's entropies [28]; see also [9]. In the vocabulary of Kruskov, the quantity (13) is called a discrepancy and typically appears in error estimates based on Kruskov's stability theory [24]. Unfortunately, the use of discrepancies does not allow for error propagation and cancellation to take place. In [26], Laforest applied Liu's wave tracing method to decompose the solution into its component waves $\mathcal{W}$ and assigned to each wave α at time t_j , a residual $R_\alpha(t_j)$ of the form given in Lemma 2.2. The following conjecture was then stated and a proof was given in the special case of initial data containing only shocks.

Conjecture 2.1 Let v be the entropy solution of the nonlinear scalar conservation law (1) where f is strictly convex and $v_0 \in L^\infty(\mathbb{R}) \cap \text{BV}(\mathbb{R})$. Then for any Δt satisfying (7), any time t_N , any sequence $\{\theta_{i,j}\}_{i,j}$ of numbers in $[-1, 1]$, and any uniform a priori bound $\epsilon > 0$ on the size of rarefaction waves, we have that the approximate solution w obtained with Glimm's scheme satisfies

$$\|v(\cdot, t_N) - w(\cdot, t_N)\|_{L^1} \leq \|v_0(\cdot) - w(\cdot, 0)\|_{L^1} + \sum_{\alpha \in \mathcal{W}} \left| \sum_{j=1}^N R_\alpha(t_j) \right| + \mathcal{O}(\epsilon^2).$$

This conjecture, for which there is ample numerical evidence [27], therefore clearly states that up to a small term of order $\mathcal{O}(\epsilon^2)$, the signed residuals of Lemma 2.2 can be used as error estimators. More importantly, it states that additive cancellation can and does occur among the residuals at the level of the waves. The $\mathcal{O}(\epsilon^2)$ term is unavoidable if error cancellation is to appear since it accounts for those (rare) cases where the absolute value of (12) is different from (13). Below, we propose an extension to systems of the definition of residual found in Lemma 2.2.

Consider a Riemann problem with a discontinuity located at the origin $x_0 = 0$ at time $t_0 = 0$ and separating two states w^- and w^+ . The solution w is formed of $n + 1$ states

$$w^- = w_0, w_1, \dots, w_n = w^+,$$

separating n waves of either shock or rarefaction type, each of which is parametrized as

$$w_k = T_k(p_k(w^-, w^+))(w_{k-1}), \quad k = 1, 2, \dots, n.$$

Suppose that, as in Glimm's scheme, a state w^* is chosen at time Δt somewhere along the family of curves joining w^- to w^+ , say in the k^* -th family

$$w^* = T_{k^*}(\sigma^*)(w_{k-1}),$$

for some $0 < \sigma^* \leq |p_{k^*}(w^-, w^+)|$.

Definition 2.1 If the k -th family contains a shock wave, then we define the residual to be

$$R_{i,j+1}^{(k)}(w) = p_k(w^-, w^+) (\Delta x \text{sign}(k - k^*) - s^{(k)}(w_{k-1}, w_k) \Delta t) \quad (14)$$

with $\text{sign}(0) = -1$. If the k -th wave is a rarefaction wave then we define

$$R_{i,j+1}^{(k)}(w) = \int_{-\Delta x}^{\Delta x} r^{(k)}(x) - r^{(k)}(\lambda^{(k^*)}(w^*) \Delta t) dx, \quad (15)$$

where

$$r^{(k)}(x) = \begin{cases} \lambda^{(k)}(w_{k-1}) & \text{if } x < x_i + \lambda^{(k)}(w_{k-1}) \Delta t, \\ \frac{x - x_i}{t - t_j} & \text{if } x \in [x_i + \lambda^{(k)}(w_{k-1}) \Delta t, x_i + \lambda^{(k)}(w_k) \Delta t], \\ \lambda^{(k)}(w_k) & \text{if } x_i + \lambda^{(k)}(w_k) \Delta t \leq x. \end{cases}$$

The quantities in Lemma 2.2 are given in terms of states while the quantities in Definition 2.1 are described with the help of wave speeds. When the scalar conservation law has a strictly convex flux, then

$$w^+ - w^- = (f')^{-1}(\xi)(f'(w^+) - f'(w^-)) = (f')^{-1}(\xi)p_1(w^-, w^+).$$

Comparing formulas (11) and (14) for shocks, we find

$$\int_{I_i \times J_{j+1/2}} \int R(w)(x, t) dx dt = (f')^{-1}(\xi) R_{i,j+1}^{(1)}(w). \quad (16)$$

For rarefactions, the expression (12) can be written

$$\int_{I_i \times J_{j+1/2}} \int R(w)(x, t) dx dt = \int_{-\Delta x}^{\Delta x} (f')^{-1}(\xi) (r^{(1)}(x) - r^{(1)}(\theta_{i,j+1} \Delta x)) dx, \quad (17)$$

which, if the mean value theorem for integrals could be used, would be equal to $(f')^{-1}(\xi) R_{i,j+1}^{(1)}(w)$.

In fact, since our objective is to measure the error with respect to the conserved variables, rather than in the coordinates $p_1, p_2, \dots, p_n$, we will always post-process the residuals in each family $R^{(1)}, \dots, R^{(n)}$ by computing

$$\Delta^{(k)} R^{(k)}, \quad (18)$$

where $\Delta^{(k)}(w^{(k-1)}) := d/d\sigma(T_k(\sigma)(w_{k-1}))$. This is the obvious extension of identities (16) and (17) relating the notion of residuals in conserved variables, as in Lemma 2.2, to the notion of residual in wave coordinates, as in Definition 2.1.

3 An Adaptive Glimm's Scheme with an Error Estimator

In this section, we describe both the error estimators and the way in which the error estimators are used to improve Glimm's scheme. The error estimator is to be an extension of the one already obtained for Glimm's scheme [26] but currently requiring Liu's cumbersome wave-tracing description.

To make the new error estimator work for Glimm's scheme implemented as a finite difference scheme, then at the $j + 1$ -st timestep, the error estimator at the mesh points (x_{i-1}, t_{j+1}) and (x_{i+1}, t_{j+1}) (with $i + j$ odd) should depend only on

- i) the states in the neighboring cells $I_{i-3}, I_{i-1}, I_{i+1}$ and I_{i+3} at time t_j ;
- ii) the values of the error estimator at the cell interfaces x_{i-2}, x_i and x_{i+2} at time t_j .

In this sense, the error estimator should be computable locally in space and time.

The previous remarks therefore suggest that we focus our attention on the update of the approximate solution at a single cell interface. For indices i and j such that $i + j$ even, consider at time t_j the two states $w_{i-1,j} := w(\cdot, t_j)|_{I_{i-1}}$, and $w_{i+1,j} := w(\cdot, t_j)|_{I_{i+1}}$, defining the Riemann problem at the mesh point (x_i, t_j) . For simplicity, we will write $w_i^- := w_{i-1,j}$ and $w_i^+ := w_{i+1,j}$. We will assume that the neighboring Riemann problems (w_{i-2}^-, w_{i-2}^+) and (w_{i+2}^-, w_{i+2}^+) have also been solved giving rise to the sequence of intermediate states

$$\begin{aligned} w_{i-2}^- &= w_{i-2}^{(0)}, w_{i-2}^{(1)}, \dots, w_{i-2}^{(n)} := w_{i-2}^+, \\ w_i^- &= w_i^{(0)}, w_i^{(1)}, \dots, w_i^{(n)} := w_i^+, \\ w_{i+2}^- &= w_{i+2}^{(0)}, w_{i+2}^{(1)}, \dots, w_{i+2}^{(n)} := w_{i+2}^+. \end{aligned}$$

For each wave separated by two states $w_i^{(k-1)}$ and $w_i^{(k)}$, we compute the characteristic speeds

$$\lambda_i^{(k)-} = \lambda^{(k)}(w_i^{(k-1)}), \quad \lambda_i^{(k)+} = \lambda^{(k)}(w_i^{(k)}).$$

For linearly degenerate fields, the characteristic speeds can be replaced by the values of some normalized parametrization of the shock curve, say arc-length.

Associated to each cell interface at time t_j and to each wave family k , there will be two types of error. The first type of error will be called the dynamic error and at the mesh point (x_i, t_j) it will be denoted

$$\mathcal{D}_{i,j}^{(k)}$$

while the second type of error will be called the static error and denoted

$$\mathcal{S}_{i,j}^{(k)}.$$

Intuitively, the static errors represent the absolute value of the sum of all the residuals of the waves that interacted and were cancelled within the solution. The dynamic errors represent the sum of the residuals for the waves that are still present within the solution. A rigorous definition will be given in the next section.

For practical purposes, the dynamic and static errors need to be translated into pointwise quantities of the original physical variables. In the spirit of the remarks made at the end of Section 2.3, the pointwise estimate of the L^1 error at a mesh point (x_i, t_j) will be given by

$$\varepsilon_{i,j} = \sum_k \left\| \Delta^{(k)}(w_i^{(k-1)}) (|\mathcal{D}_{i,j}^{(k)}| + \mathcal{S}_{i,j}^{(k)}) \right\| \quad (19)$$

while the estimate of the $L^1(\mathbb{R})$ error in space at time $t = t_j$ will be

$$\mathcal{E}_j = \sum_i \varepsilon_{i,j}. \quad (20)$$

3.1 Error generation, propagation and cancellation

Our basic assumption is that error propagation and cancellation is a roughly linear process. This is identical to the process which Glimm showed was responsible for nonlinear wave propagation when the total variation of the solution was small [15]. We also insist that error propagation should be a conservative process.

Assume that the updated constant state $w^* := w_{i,j+1}$ of w on the interval I_i at time t_j is known. Since the state w^* belongs to the curve joining w_i^- and w_i^+ , then there exists a k^* such that all the waves

$$\sigma_i^{(1)-} := p_1(w_i^-, w_i^+), \dots, \sigma_i^{(k^*-1)-} := p_{k^*-1}(w_i^-, w_i^+), \quad (21)$$

travel to the left, i.e., contribute to the new Riemann problem at $x_{i-1,j+1}$, the waves

$$\sigma_i^{(k^*)-} := p_{k^*}(w_i^-, w^*) \text{ and } \sigma_i^{(k^*)+} := p_{k^*}(w^*, w_i^+), \quad (22)$$

travel respectively to the left and the right, and finally

$$\sigma_i^{(k^*+1)+} := p_{k^*+1}(w_i^-, w_i^+), \dots, \sigma_i^{(n)+} := p_n(w_i^-, w_i^+), \quad (23)$$

travel to the right. If the k^* -th wave initially generated at $x_{i,j}$ was a shock wave, then only one of the two waves in (22) can possibly appear. On the other hand, if it is a rarefaction, then it can be split into two smaller rarefactions. For each one of these two rarefaction waves, the scheme induces an error because the waves are obliged to travel a distance $\pm \Delta x$ in a period of time Δt . The error associated to each rarefaction wave therefore needs to be distributed among the nodes $x_{i-1,j+1}$ and $x_{i+1,j+1}$.

For the k waves in (21), we define the left and right travelling local residuals

$$\mathcal{R}_{i,j}^{(k)-} = R_{i,j+1}^{(k)}(w), \quad \mathcal{R}_{i,j}^{(k)+} = 0.$$

For the k waves in (23), we define

$$\mathcal{R}_{i,j}^{(k)-} = 0, \quad \mathcal{R}_{i,j}^{(k)+} = R_{i,j+1}^{(k)}(w).$$

Finally, if the k^* wave initially at $x_{i,j}$ was a shock then we write

$$\mathcal{R}_{i,j}^{(k^*)\pm} = R_{i,j+1}^{(k^*)}(w), \quad \mathcal{R}_{i,j}^{(k^*)\mp} = 0,$$

with $(+)$ if the wave travels to the right and $(-)$ if it travels to the left. For rarefactions, we split the residual into left and right travelling contributions

$$\begin{aligned} \mathcal{R}_{i,j}^{(k^*)-} &= \int_{-\Delta x}^{\lambda^{(k^*)}(w^*)\Delta t} r^{(k)}(x) - r^{(k)}(\lambda^{(k^*)}(w^*)\Delta t) dx, \\ \mathcal{R}_{i,j}^{(k^*)+} &= \int_{\lambda^{(k^*)}(w^*)\Delta t}^{\Delta x} r^{(k)}(x) - r^{(k)}(\lambda^{(k^*)}(w^*)\Delta t) dx. \end{aligned}$$

Given that the residuals travel in this manner, one would like to define similar contributions to the errors at $x_{i\pm 1,j+1}$ for the dynamic and static errors. The natural choice is define the ratio of the strength of the waves travelling left or right

$$\nu_{i,j}^{(k)-} := \frac{p_k(w_i^-, w^*)}{p_k(w_i^-, w_i^+)}, \quad \nu_{i,j}^{(k)+} := \frac{p_k(w^*, w_i^+)}{p_k(w_i^-, w_i^+)},$$

and use these to split the dynamic and static errors among the nodes $x_{i-1,j+1}$ and $x_{i+1,j+1}$ according to

$$\begin{aligned}\mathcal{D}_{i,j}^{(k)-} &:= \nu_{i,j}^{(k)-} \mathcal{D}_{i,j}^{(k)}, & \mathcal{D}_{i,j}^{(k)+} &:= \nu_{i,j}^{(k)+} \mathcal{D}_{i,j}^{(k)}, \\ \mathcal{S}_{i,j}^{(k)-} &:= \nu_{i,j}^{(k)-} \mathcal{S}_{i,j}^{(k)}, & \mathcal{S}_{i,j}^{(k)+} &:= \nu_{i,j}^{(k)+} \mathcal{S}_{i,j}^{(k)}.\end{aligned}$$

This completes our description of the process of propagation of errors. Notice that no loss occurred in the propagation of either the dynamic or the static error.

At this point, it would be natural to add together the errors $\mathcal{R}_{i-2,j}^{(k)+}$, $\mathcal{D}_{i-2,j}^{(k)+}$, $\mathcal{R}_{i,j}^{(k)-}$ and $\mathcal{D}_{i,j}^{(k)-}$ in order to construct the error at $x_{i-1,j+1}$. When cancellation does not occur, that is when the incoming k waves are of the same type, then we propose to define

$$\mathcal{D}_{i-1,j+1}^{(k)} = \mathcal{R}_{i-2,j}^{(k)+} + \mathcal{D}_{i-2,j}^{(k)+} + \mathcal{R}_{i,j}^{(k)-} + \mathcal{D}_{i,j}^{(k)-}, \quad (24)$$

$$\mathcal{D}_{i+1,j+1}^{(k)} = \mathcal{R}_{i,j}^{(k)+} + \mathcal{D}_{i,j}^{(k)+} + \mathcal{R}_{i+2,j}^{(k)-} + \mathcal{D}_{i+2,j}^{(k)-}, \quad (25)$$

$$\mathcal{S}_{i-1,j+1}^{(k)} = \mathcal{S}_{i-2,j}^{(k)+} + \mathcal{S}_{i,j}^{(k)-}, \quad (26)$$

$$\mathcal{S}_{i+1,j+1}^{(k)} = \mathcal{S}_{i,j}^{(k)+} + \mathcal{S}_{i+2,j}^{(k)-}. \quad (27)$$

On the other hand, when a k -shock meets a k -rarefaction, say at $x_{i-1,j+1}$, then the amount of cancellation will be

$$\mu_{i-1,j+1}^{(k)} = \frac{1}{2} \left(|\sigma_{i-2,j}^{(k)+}| + |\sigma_{i,j}^{(k)-}| - |\sigma_{i-2,j}^{(k)+} + \sigma_{i,j}^{(k)-}| \right).$$

This suggests that when cancellation occurs, then we retain only a fraction of the errors in (24)–(27) where the fraction is equal to the fraction of the wave strength that has yet to interact, namely

$$\begin{aligned}\mathcal{D}_{i-1,j+1}^{(k)} &= \left(1 - \frac{\mu_{i-1,j+1}^{(k)}}{p_k(w_{i-2}^*, w_{i-2}^+)} \right) (\mathcal{R}_{i-2,j}^{(k)+} + \mathcal{D}_{i-2,j}^{(k)+}) \\ &\quad + \left(1 - \frac{\mu_{i-1,j+1}^{(k)}}{p_k(w_i^-, w_i^*)} \right) (\mathcal{R}_{i,j}^{(k)-} + \mathcal{D}_{i,j}^{(k)-}),\end{aligned} \quad (28)$$

$$\begin{aligned}\mathcal{S}_{i-1,j+1}^{(k)} &= \frac{\mu_{i-1,j+1}^{(k)}}{p_k(w_{i-2}^*, w_{i-2}^+)} |\mathcal{R}_{i-2,j}^{(k)+} + \mathcal{D}_{i-2,j}^{(k)+}| + \mathcal{S}_{i-2,j}^{(k)+} \\ &\quad + \frac{\mu_{i-1,j+1}^{(k)}}{p_k(w_i^-, w_i^*)} |\mathcal{R}_{i,j}^{(k)-} + \mathcal{D}_{i,j}^{(k)-}| + \mathcal{S}_{i,j}^{(k)-}.\end{aligned} \quad (29)$$

Obvious extensions of $\mu_{i+1,j+1}^{(k)}$, $\mathcal{D}_{i+1,j+1}^{(k)}$ and $\mathcal{S}_{i+1,j+1}^{(k)}$ can be defined. We remark that the dynamic error has a sign while the static error is always positive.

3.2 An optimal sampling in Glimm's choice

The title of this section is slightly a misnomer since we do not provide a proof that our proposed sampling is optimal. In fact, several choices were tested and the one which we judged to be the most reliable numerically is the one which we present.

The first step is to estimate the $n + 1$ different errors which would be generated if the sampled value w^* was chosen among the intermediate states

$$w_i^- = w_i^{(0)}, w_i^{(1)}, \dots, w_i^{(n)} := w_i^+,$$

in the solution of the Riemann problem joining w_i^- and w_i^+ . First of all, we note that it is clearly sufficient to use the values of the dynamic and static error at time t_j without computing the updated values at time t_{j+1} because propagation of error is conservative. Although this is not consistent without our definition of local error (19), the error in the $n + 1$ choices is computed with the expression

$$\varepsilon_{i,j} = \sum_k \left\| \Delta^{(k)}(w_i^{(k-1)}) \left(\mathcal{D}_{i,j}^{(k)} + \mathcal{R}_{i,j}^{(k)-} + \mathcal{R}_{i,j}^{(k)+} \right) \right\|. \quad (30)$$

Notice that because we are sampling at the intermediate states, there cannot be cancellation between $\mathcal{R}_{i,j}^{(k)-}$ and $\mathcal{R}_{i,j}^{(k)+}$ (one of them has to vanish).

Suppose that the two lowest values of the error (30) are those which occur when sampling at $w_i^{(k-1)}$ and at $w_i^{(k)}$. If these states are joined by a shock, then we pick w^* equal to the state for which the error was least. If the two states join a rarefaction wave, then we pick the state w^* that minimizes the total amount of residual generated

$$\left| \mathcal{R}_{i,j}^{(k)-} \right| + \left| \mathcal{R}_{i,j}^{(k)+} \right|. \quad (31)$$

One notes that the choice of w^* is independent of either the dynamic or the static errors. In practice, this has been found to be the best choice, although not the only choice. In fact, it tends to introduce too much “diffusion” in the sense that rarefaction waves tend to spread apart more quickly at the edges of the rarefaction, see Figure 4.

4 Numerical Results

In this section, we present initial data supporting our two main claims. These claims are that our adaptive version of Glimm’s scheme is i) more accurate than other versions of Glimm’s scheme and ii) possesses an accurate local and global error estimate of the error in L^1 for all time. For ii) to hold, the error estimator must account for error propagation and cancellation.

In the numerical experiments studied below, Glimm’s scheme was run using a fixed constant timestep throughout. This simplified the task of comparing it’s solution to highly resolved solutions obtained using a weighted essentially non-oscillatory (WENO) scheme described in [22, 39]. The error estimator was computed at each timestep and the true error was estimated by comparing pointwise the difference between Glimm’s coarse solution w and a highly resolved WENO solution v . More specifically, the pointwise computed error was the following function of x computed for a fixed T ,

$$|v(x, T) - w(x, T)|.$$

The L^1 computed error was the following function of t

$$\int_{\mathbb{R}} |v(x, t) - w(x, t)| dx.$$

4.1 Burgers’ equation

The inviscid Burger’s equation is classic example of a nonlinear scalar conservation law. It is defined by it’s strictly convex flux which is simply $f(u) = u^2/2$. For the tests presented below, we used the residuals defined in Lemma 2.2 rather than post-processing those of Definition 2.1

by multiplication by $\Delta^{(1)}$. The problems chosen allow us to study problems where shocks interact with rarefactions, i.e., with cancellation, and also smooth periodic problems where shocks are formed.

4.1.1 Initial data with 6 shocks and 6 rarefactions

The initial data is symmetric within the domain $[-10, 10]$ and solved up to a time $t = 0.5$ before any of the waves exit the domain. We consider the initial data

$$w(x, 0) = \begin{cases} 0 & \text{for } x \in [-10, -4.074) \cup [4.074, 10], \\ -5.5 & \text{for } x \in [-4.074, -3.333), \\ -3 & \text{for } x \in [-3.333, -2.592), \\ -4 & \text{for } x \in [-2.592, -1.851), \\ -1 & \text{for } x \in [-1.851, -1.111), \\ -2 & \text{for } x \in [-1.111, -0.370), \\ 0 & \text{for } x \in [-0.370, 0.370), \\ 2 & \text{for } x \in [0.370, 1.111), \\ 1 & \text{for } x \in [1.111, 1.851), \\ 4 & \text{for } x \in [1.851, 2.592), \\ 3 & \text{for } x \in [2.592, 3.333), \\ 5.5 & \text{for } x \in [3.333, 4.074), \end{cases} \quad (32)$$

containing 6 shocks and 6 rarefactions. The reference solution computed using WENO required 3456 subintervals. Figure 1 shows the final solutions at time $t = 0.5$.

In Figure 2, we notice that the position and strength of the pointwise error is well approximated by the quantities $\varepsilon_{i,j}$, defined by formula (19). Note also that the errors are symmetric with respect to the origin, as one would expect. Figure 3 demonstrates that the estimate of the L^1 error, $\mathcal{E}_j$ is quite accurate, even for long periods of time.

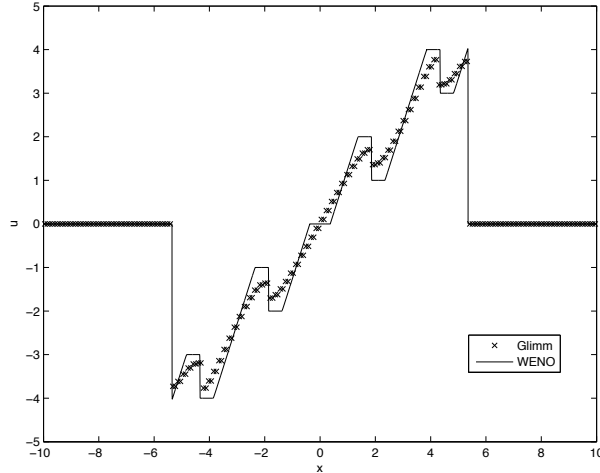


Fig.1 For the problem (32), Glimm's scheme with 108 subintervals and a resolved solution using WENO. The solutions are presented at time $t = 0.5$.

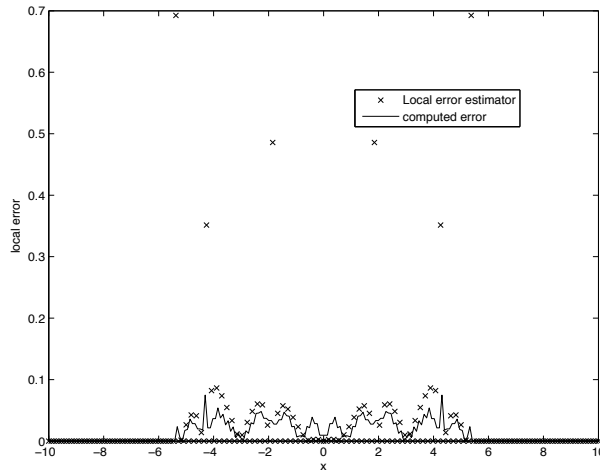


Fig.2 For the problem (32), the error estimator $\varepsilon_{i,j}$ and the computed error, as a function of x for a fixed time $t = 0.5$. Glimm's scheme is solved using 108 subintervals.

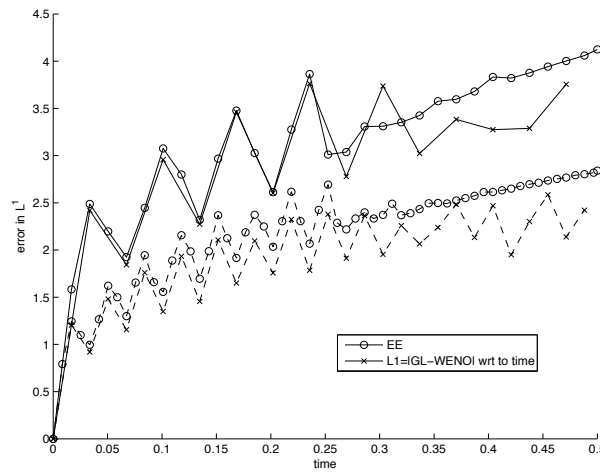


Fig.3 For the problem (32), the error estimator $\mathcal{E}_j$ and the $L^1(\mathbb{R})$ norm of the computed error, both presented as functions of t . The solid lines are obtained using Glimm's scheme and 108 subintervals while the dashed lines are obtained using 216 subintervals.

4.2 Euler's system for ideal gas dynamics

Euler's equations for an ideal compressible gas in Eulerian coordinates in 1D are

$$\begin{aligned} \rho_t + (\rho v)_x &= 0, \\ (\rho v)_t + (\rho v^2 + p)_x &= 0, \\ \left(\rho\left(\frac{1}{2}v^2 + e\right)\right)_t + \left(\rho v\left(\frac{1}{2}v^2 + e\right) + pv\right)_x &= 0, \end{aligned} \quad (33)$$

where ρ is density, v is velocity, p is pressure, e is internal energy, the equation of state is $e = p/(\rho(\gamma - 1))$ and γ is 1.4. The first and third family of waves is genuinely nonlinear and the 2nd family is linearly degenerate and describes only contact discontinuities. For contact discontinuities, the strength of the wave is measured by the strength of the jump in density across the shock.

For the Euler system, the error is estimated in wave coordinates and therefore, it is necessary to post-process the error in (19) using the derivatives $\Delta^{(k)}(w^{(k-1)}) := \frac{d}{d\sigma}(T_k(\sigma)(w_{k-1}))$. Simple calculations show that if $\Delta_1 v := (u^+ - c^+) - (u^- - c^-)$ and $\Delta_3 v := (u^+ + c^+) - (u^- + c^-)$, then

$$\Delta^{(1)} = \begin{pmatrix} -\frac{\rho^-}{c^-} \frac{2}{\gamma+1} \\ \frac{2}{\gamma+1} \\ -\frac{p^-}{c^-} \frac{2\gamma}{\gamma+1} \end{pmatrix} \Delta_1 v \quad \Delta^{(3)} = \begin{pmatrix} \frac{\rho^-}{c^-} \frac{2}{\gamma+1} \\ \frac{2}{\gamma+1} \\ \frac{p^-}{c^-} \frac{2\gamma}{\gamma+1} \end{pmatrix} \Delta_3 v. \quad (34)$$

The vector $\Delta^{(2)}$ is simply $(0, 1, 0)^T$.

4.2.1 Sod's problem

The so-called Sod problem is a shock tube problem, i.e., a Riemann problem, with an initial jump at $x = 0.3$. The initial data is

$$(\rho, v, p)(x, 0) = \begin{cases} (1, 0.75, 1) & \text{for } x \in [0, 0.3), \\ (0.125, 0, 0.1) & \text{for } x \in [0.3, 1.0]. \end{cases} \quad (35)$$

The solution is formed of the left moving rarefaction, a contact discontinuity and a right moving shock wave, see Figure 4. For this problem, an analytical solution is available and but was not used to measure the error. Instead, the approximate solution obtained by our WENO scheme was used. The solution was solved up to time $t = 0.2$ which is before any wave reaches the boundary of the domain.

Figure 5 shows that the error estimator identifies the location and the approximate size of the error in both the contact discontinuity and the shock wave. It also estimates quite well the error in density and in pressure across the rarefaction wave but overestimates the velocity error. In Figure 6, the sum of the L^1 errors in all the variables is again accurately estimated for a long period of time. In this problem, there is no wave interaction and cancellation so this is not the most difficult test problem. Nonetheless, the results are quite accurate and encouraging.

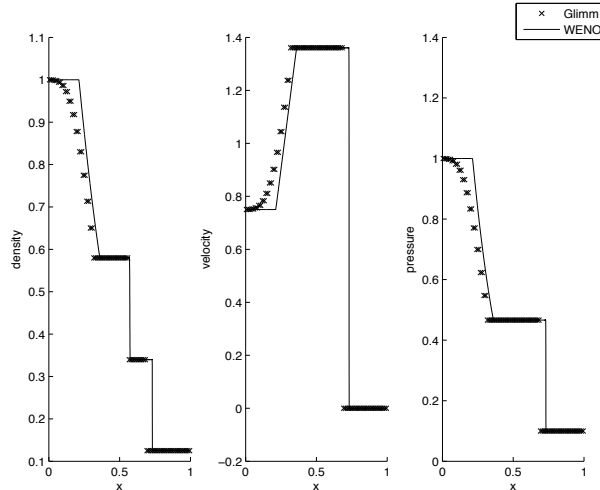


Fig.4 For Sod's problem, Glimm's scheme with 40 subintervals and a resolved solution using WENO. The solutions are presented at time $t = 0.2$.

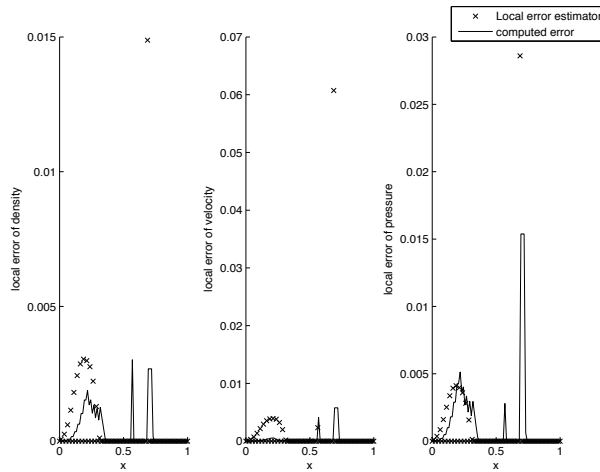


Fig.5 For Sod's problem, the error estimator $\varepsilon_{i,j}$ and the computed error, as a function of x for a fixed time $t = 0.2$. Glimm's scheme is solved using 40 subintervals.

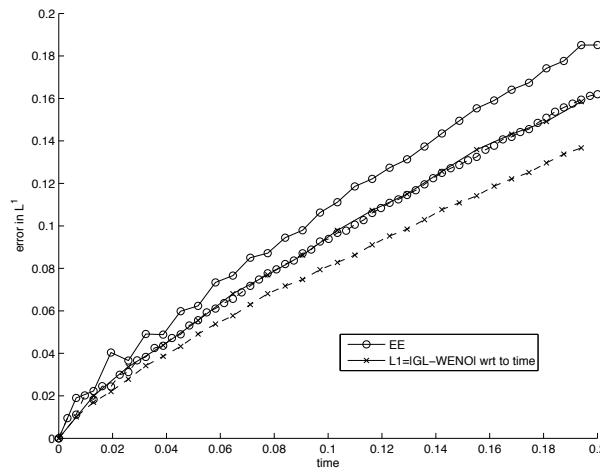


Fig.6 For Sod's problem, the error estimator $\mathcal{E}_j$ and the $L^1(\mathbb{R})$ norm of the computed error, both presented as functions of t . The solid lines are obtained using Glimm's scheme and 40 subintervals while the dashed lines are obtained using 80 subintervals.

4.2.2 An acoustic wave through a strong shock

This is the so-called Shu-Tadmor test problem [31] with initial data formed of a single strong shock with an "acoustic" wave on the right, as in Figure 7. For $\epsilon = 0.2$, the initial data is

$$(\rho, v, p)(x, 0) = \begin{cases} (3.857143, 2.629369, 10.3333) & \text{for } x \in [-10, -4), \\ (1 + \epsilon \sin(5x), 0, 1) & \text{for } x \in [-4, 10]. \end{cases} \quad (36)$$

The problem is solved up to time $t = 0.2$ and absorbing boundary conditions are applied. The reference solution was computed on 3200 subintervals.

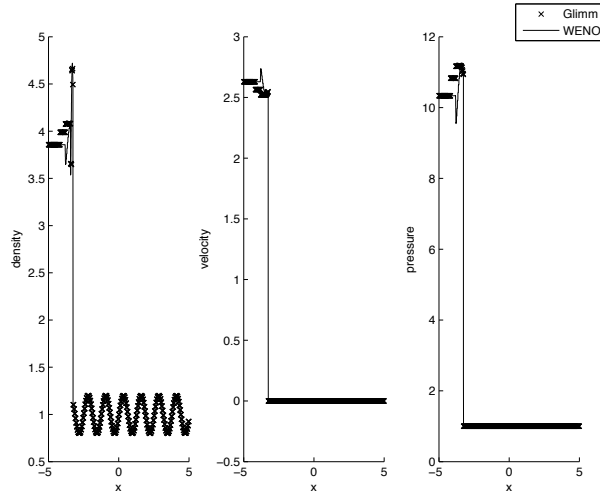


Fig.7 For the Shu-Tadmor problem, Glimm's scheme with 400 subintervals and a resolved solution using WENO. The solutions are presented at time $t = 0.2$.

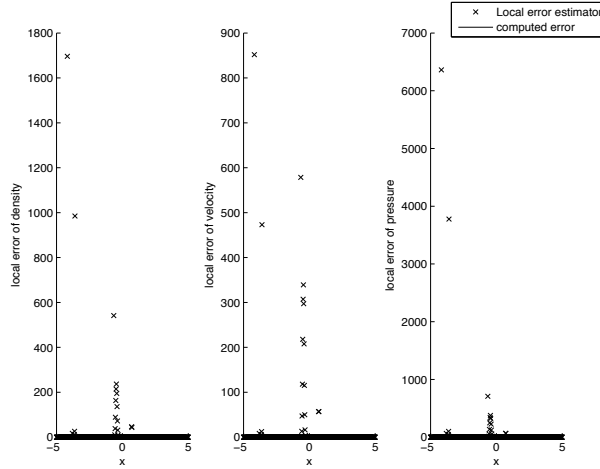


Fig.8 For the Shu-Tadmor problem, the error estimator $\varepsilon_{i,j}$ and the computed error, as a function of x for a fixed time $t = 0.2$. Glimm's scheme is solved using 400 subintervals.

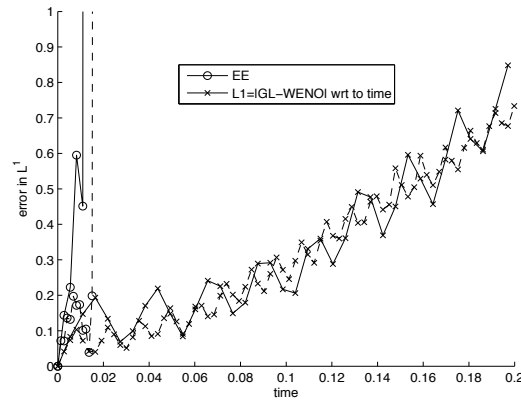


Fig.9 For the Shu-Tadmor problem, the error estimator $\mathcal{E}_j$ and the $L^1(\mathbb{R})$ norm of the computed error, both presented as functions of t . The solid lines are obtained using Glimm's scheme and 400 subintervals while the dashed lines are obtained using 800 subintervals.

The Shu-Tadmor test problem is by far the most difficult problem we have considered. Unfortunately, the error estimator overestimates the error by nearly two orders of magnitude, see Figure 9. Oscillations in the solution lead to cancellation, i.e., $\mu_{i-1,j+1}^{(k)} \neq 0$, and therefore a steady increase in the static error. This case needs further analysis.

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