An a posteriori error estimate for Glimm's scheme

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We present an a posteriori error bound for Glimm's approximate solutions [2] to nonlinear scalar conservation laws containing only shock waves, that is

$$\begin{aligned} v_t + f(v)_x &= 0, \\ v(\cdot, 0) &= v_0(\cdot) \in L^\infty(\mathbb{R}) \quad \text{and decreasing.} \end{aligned} \tag{1}$$

Using Liu's wave-tracing method [9], we show that the L^1 norm of the error is bounded by a sum of residuals containing independent contributions from each wave in the approximate solution but allowing for error cancellation among waves. The proof can also be viewed as an explicit form of a construction of Hoff and Smoller [5]. This paper contains an abbreviated description of the proof found in [8] and new numerical evidence that the error estimate should continue to hold for arbitrary L^∞ initial data.

The objective of this paper is to present new error estimators for approximate solutions that may eventually be used to build efficient adaptive schemes. For conservation laws (1), adaptive schemes are important because it is particularly difficult to accurately solve the problem in the presence of smooth and discontinuous waves. These waves are an intrinsic part of the physical process modeled by conservation laws, namely bottlenecks in vehicular traffic or waves in channels.

As an a posteriori error estimate, this result is of interest since it shows explicitly that the errors are created, propagated, and cancelled at the level of waves. This estimate contrasts with those based on Kruskov's stability theory, such as [6, 4], that don't account for such processes or with those based on the adjoint formulation which are currently limited to linear (or linearized) problems, see [1] for a survey. As a stability result, this approach might be useful to treat problems requiring an analysis in $\mathrm{BV}_{\mathrm{local}}$ such as when the initial data is periodic or in L^∞.

In Section 1 we review Glimm's scheme and introduce the local error estimator. In Section 2, we present the main results and show how certain local estimates are used to demonstrate the global estimate. Section 3 contains numerical results that show that the estimates are, in some sense, optimal and that they probably continue to hold for arbitrary initial data.

1 Glimm's scheme

Glimm's scheme, coupled with Liu's wave-tracing method, plays a central role in the qualitative study of hyperbolic systems of conservation laws. It is well-known that if f is strictly convex, then there exists a *unique* weak solution $v : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ of (1) satisfying a physical entropy condition.

We now describe Glimm's scheme. Assume $v_0 \in L^\infty(\mathbb{R})$ is decreasing and pick a discretization $\Delta x, \Delta t$ satisfying the *CFL condition*, $\sup_v |f'(v)| \leq \Delta x/\Delta t$, where the sup is over the range of v. Begin by approximating v_0 by a piecewise constant and decreasing $w(\cdot, 0)$ such that *i)* $w(\cdot, 0)$ is piecewise constant over $I_m \doteq [(m-1)\Delta x, (m+1)\Delta x]$, m even, and *ii)* $\|v_0(\cdot) - w(\cdot, 0)\|_{L^1} = \mathcal{O}(\Delta x)$. Writing $t_n \doteq n\Delta x$, the entropy solution over $\mathbb{R} \times [t_0, t_1)$ is a sequence of non-interacting discontinuities, each separated by different constant left and right hand states u^l, u^r, and travelling at the *Rankine-Hugoniot* speed

$$S(u^l, u^r) \doteq \frac{f(u^l) - f(u^r)}{u^l - u^r}. \tag{2}$$

In order to make $w(\cdot, t_1)$ into a piecewise constant function over the grid I_m with m odd, we pick a random number $\theta_1 \in [-1, 1]$ and set

$$w(\cdot, t_1)|_{I_m} \doteq \lim_{t \to t_1 -} w\big((m+1)\Delta x + \theta_1 \Delta x, t\big).$$

The procedure can be repeated indefinitely using a random number θ_{n+1} to propagate a piecewise constant approximation $w(\cdot, t_n)$ to a piecewise constant approximation $w(\cdot, t_{n+1})$. Glimm showed that for almost all random sequences $\{\theta_n\}$, w converges to v as $\Delta x \to 0$.

We summarize Liu's so-called *wave-tracing* description [9], as it applies to scalar conservation laws with decreasing v_0, which describes w as a linear superposition of discrete *waves* propagating and interacting nonlinearly.

Theorem 1. *[9] Given a random sequence $\{\theta_n\}_{n \in \mathbb{N}}$, and a region $\mathbb{R} \times [0, T]$, Glimm's approximate solution w to (1) with decreasing $w(\cdot, 0)$ can be described as a family of waves $\mathcal{W}$, where each wave $\alpha \in \mathcal{W}$ has the following characteristics,*

i) two constant left and right hand states w^l_α and w^r_α,
ii) a strength $\sigma_\alpha \doteq w^r_\alpha - w^l_\alpha$,
iii) and a position $x_\alpha(t) \in \mathbb{R}$ that satisfies $x_\alpha(t_{n+1}) - x_\alpha(t_n) = \pm\Delta x$.

The approximate solution is then constructed as

$$w(x, t_n) = w(-\infty, 0) + \sum_{\{\alpha | x_\alpha(t_n) \leq x\}} \sigma_\alpha. \tag{3}$$

We briefly explain how the waves are defined. To each discontinuity in the initial data, we associate one unique (shock) wave α. We let $x_\alpha(0)$ be the location of that discontinuity at time t_0, and set

$$w_\alpha^l \doteq \lim_{x\to x_\alpha(0)-} w(x,0), \qquad w_\alpha^r \doteq \lim_{x\to x_\alpha(0)+} w(x,0). \tag{4}$$

If we know it's position at time t_n, then we can compute

$$w_\alpha^-(t_n) \doteq \lim_{x\to x_\alpha(t_n)-} w(x,t_n), \qquad w_\alpha^+(t_n) \doteq \lim_{x\to x_\alpha(t_n)+} w(x,t_n). \tag{5}$$

and thereby define

$$x_\alpha(t) = x_\alpha(t_n) + (t-t_n)S\big(w_\alpha^-(t_n), w_\alpha^+(t_n)\big), \tag{6}$$

$$x_\alpha(t_{n+1}) = x_\alpha(t_n) + \Delta x \operatorname{sign}\big(S\big(w_\alpha^-(t_n), w_\alpha^+(t_n)\big)\Delta t - \theta_{n+1}\Delta x\big). \tag{7}$$

The family of waves $\mathcal{W}$ has a natural ordering. A wave α is said to be smaller than a wave β, written $\alpha < \beta$, if $x_\alpha(0) < x_\beta(0)$, or when equality occurs, if $w_\alpha^l > w_\beta^l$.

Let u be the entropy solution of (1) with initial data $w(\cdot,0)$. Then at each time t, we can assign a position $y_\alpha(t)$ to the wave α in u by setting $y_\alpha(0) = x_\alpha(0)$ and, at each time t, making it equal to the position of the discontinuity in u at time t with which the initial discontinuity interacted.

If w is an approximate solution generated by Glimm's scheme then the residual is, in the sense of distributions, $w_t + f(w)_x$. Detailed arguments, found in [7, 8], show that there exists a well-defined notion of residuals that can be computed a posteriori and assigned uniquely to each wave in Liu's decomposition.

Definition 1. *Given a wave α at time t_n and $s = S(w_k^-(t_n-), w_\alpha^+(t_n-))$, then the residual is*

$$R(\alpha, t_n) \doteq \sigma_\alpha\big(\Delta x \operatorname{sign}(s\Delta t - \theta_n \Delta x) - s\Delta t\big). \tag{8}$$

2 Main Result

In this section we present a brief proof of the following Theorem. The proof depends on Lemmas 1-4 whose proofs can be found in [8].

Theorem 2. *Consider decreasing initial data $v_0 \in L^\infty(\mathbb{R})$ to (1). Suppose the approximation $w(\cdot,0)$ to the initial data contains only shocks and $v_0 - w(\cdot,0) \in L^1(\mathbb{R})$. Then for any Δt satisfying the CFL condition, any time t, and any sequence $\{\theta_k\}_k$ of numbers in $[-1,1]$, we have that the approximate solution w obtained with Glimm's scheme satisfies*

$$\big\|v(\cdot,t) - w(\cdot,t)\big\|_{L^1(\mathbb{R})} \le \big\|v_0(\cdot) - w(\cdot,0)\big\|_{L^1(\mathbb{R})} + \sum_{\alpha\in\mathcal{W}} \left| \sum_{k=1}^{\lfloor t/\Delta t\rfloor} R(\alpha, t_k)\right|. \tag{9}$$

This estimate contrasts with previous error bounds for finite difference schemes, such as those of Kröner and Ohlberger [6] or Gosse and Makridakis [4], that are of the form $\sum_\alpha \sum_k |R(\alpha, t_k)|$. We emphasize that this error bound is free of unknown constants, valid for all random sequences $\{\theta_k\}$, and accounts for error propagation and cancellation among waves.

Rather than compare the waves in w with those in v, we first compare them to those in the entropy solution u defined by the initial data $w(\cdot, 0)$.

Definition 2. *If the waves in $\mathcal{W}$ are ordered $\mathcal{W} = [\alpha(1), \dots, \alpha(N)]$, then we call $X(t) = [x_{\alpha(i)}(t)]_{i=1}^N = [x_{\alpha(1)}(t), \dots, x_{\alpha(N)}(t)]^T$ and $Y(t) = [y_{\alpha(i)}(t)]_{i=1}^N$, the trajectories of shock waves in respectively w and u.*

Given two sets of trajectories $Z(t) = [z_\alpha(t)]_{\alpha\in\mathcal{W}}$ and $\widetilde{Z}(t) = [\widetilde{z}_\alpha(t)]_{\alpha\in\mathcal{W}}$, we define the discrepancy as $d\big(\widetilde{Z}(t), Z(t)\big) \doteq \big[|\sigma_\alpha| \cdot \big|\widetilde{z}_\alpha(t) - z_\alpha(t)\big|\big]_{\alpha\in\mathcal{W}}$.

Definition 3. *We define the continuous trajectories $X^{(0)}(t) = [x_\alpha^{(0)}(t)]_{\alpha\in\mathcal{W}}$ by requiring that the position $x_\alpha^{(0)}(t)$ of each wave α satisfy $x_\alpha^{(0)}(0) = x_\alpha(0)$ and $\dot{x}_\alpha^{(0)}(t) = S\big(w_\alpha^-(t), w_\alpha^+(t)\big)$, for all positive t.*

Lemma 1. *If $u_0(\cdot) = w(\cdot, 0)$ is decreasing and $\mathbf{1}^T = [1, 1, \dots, 1]$, then*

$$\big\|u(\cdot, t) - w(\cdot, t)\big\|_{L^1(\mathbb{R})} = \mathbf{1}^T d\big(Y(t), X(t)\big), \qquad \forall t > 0. \tag{10}$$

Lemma 2. *If $\alpha \in \mathcal{W}$ is a shock wave then for all $t \geq 0$, the α-th component of $d(X^{(0)}(t), X(t))$ satisfies*

$$|\sigma_\alpha| \cdot \big|x_\alpha^{(0)}(t) - x_\alpha(t)\big| = \left|\sum_{k=1}^{\lfloor t/\Delta t\rfloor} R(\alpha, t_k)\right|. \tag{11}$$

The first lemma relates the L^1 norm to discrepancies between Y and X while the second lemma relates the residuals to discrepancies between $X^{(0)}$ and X. The problem is now to find a sequence of corrections that, when applied to $X^{(0)}$, provide the exact trajectories Y yet preserve the total quantity of residuals $\mathbf{1}^T R(t) = \mathbf{1}^T d(X^{(0)}(t), X(t))$.

Definition 4. *Consider a consecutive set of waves $\mathcal{F} = [\alpha(1), \dots, \alpha(n)]$. The free trajectories of $\mathcal{F}$ are the trajectories $F(t) = [f_{\alpha(i)}(t)]_{i=1}^n$ of the discontinuities in the solution $V(x, t)$ to the conservation law (1) with initial data*

$$V(x, 0) = \begin{cases} w_{\alpha(1)}^l & \text{if } x \leq x_{\alpha(1)}(0), \\ w_{\alpha(i)}^r & \text{if } x_{\alpha(i)}(0) < x \leq x_{\alpha(i+1)}(0) \text{ and } i \in \{1, \dots, n-1\}, \\ w_{\alpha(n)}^r & \text{if } x_{\alpha(n)}(0) < x. \end{cases} \tag{12}$$

We will say that an interaction occured in F at time t_* if it occured in the entropy solution V at time t_*.

Definition 5. *A matrix C is conservative if $\mathbf{1}^T C = \mathbf{1}^T$.*

The following definition provides, for each fixed time t_*, the two trajectories $X^{(1)}$ and $X^{(2)}$. The Lemmas 3 and 4 then indicate how to relate Y and $X^{(2)}$ and finally to $X^{(1)}$.

Definition 6. *For a fixed time t_*, let $\mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(m)}$ be the set of waves forming the m discontinuities in w at time t_*. For each set $\mathcal{F}^{(i)}$, suppose the free trajectories $F^{(i)}(t)$ associated to $\mathcal{F}^{(i)}$ posses n_i discontinuities at time $t = t_*$ and suppose these discontinuities form the sets $\mathcal{S}_{(i,1)}, \ldots, \mathcal{S}_{(i,n_i)}$. Let F be the trajectories defined by*

$$F(t) = \left[F^{(1)}(t)^T, \cdots, F^{(m)}(t)^T\right]^T, \tag{13}$$

and construct the trajectories $X^{(1)}$ satisfying

$$\begin{aligned} X^{(1)}(t) &= F(t), \qquad && \forall t \in [0, t_*], \\ \dot{X}^{(1)}(t) &= \dot{X}(t), \qquad && \forall t > t_*. \end{aligned} \tag{14}$$

Let T_* be the next time at which either an interaction occurs in w, or in one of the free trajectories $F^{(1)}, \ldots, F^{(m)}$. Crossings when $x_\alpha^{(1)}(t) = x_\beta^{(1)}(t)$ but $\alpha \in \mathcal{F}^{(i)}$ and $\beta \in \mathcal{F}^{(j)}$, $i \neq j$, are not considered interactions. Define the trajectories $X^{(2)}$ by

$$\begin{aligned} X^{(2)}(t) &= F(t), \qquad && \forall t \in [0, T_*], \\ \dot{X}^{(2)}(t) &= \dot{X}(t), \qquad && \forall t > T_*. \end{aligned} \tag{15}$$

Lemma 3. *For each fixed time t_*, there exists a conservative matrix $B(t)$ such that*

$$d\big(X^{(2)}(t), X(t)\big) \le B(t) \cdot d\big(X^{(1)}(t), X(t)\big), \qquad \forall t. \tag{16}$$

Lemma 4. *For each fixed time t_*, there exists a conservative matrix $C(t)$ such that*

$$d\big(Y(t), X(t)\big) \le C(t) \cdot d\big(X^{(2)}(t), X(t)\big), \qquad \forall t \le T_*. \tag{17}$$

We now sketch the proof of Theorem 2.

Proof (of Theorem 2). The proof proceeds by induction. The induction hypothesis is that at each time t_*, there exists a conservative matrix $A(t)$ such that

$$d\big(X^{(1)}(t), X(t)\big) \le A(t) \cdot d\big(X^{(0)}(t), X(t)\big), \qquad \forall t. \tag{18}$$

We begin by showing that if the induction hypothesis holds at time t_*, then the theorem holds for all $t \in [t_*, T_*]$. Applying in the following order Lemma 1, Lemma 4, Lemma 3, the induction hypothesis (18), Definition 5, and Lemma 2, we obtain

$$
\begin{aligned}
\big\|u(\cdot,t)-w(\cdot,t)\big\|_{L^1(\mathbb{R})} &= \mathbf{1}^T d\big(Y(t),X(t)\big)\\
&\le \mathbf{1}^T C(t)\cdot d\big(X^{(2)}(t),X(t)\big)\\
&\le \mathbf{1}^T C(t)B(t)\cdot d\big(X^{(1)}(t),X(t)\big)\\
&\le \mathbf{1}^T C(t)B(t)A(t)\cdot d\big(X^{(0)}(t),X(t)\big)\\
&= \sum_{\alpha\in\mathcal{W}}\left|\sum_{k=1}^{\lfloor t/\Delta t\rfloor} R(\alpha,t_k)\right|. \qquad (19)
\end{aligned}
$$

To prove estimate (9), it now suffices to apply the triangle inequality on $\|v(\cdot,t)-u(\cdot,t)+u(\cdot,t)-w(\cdot,t)\|_{L^1}$ and the fact that the evolution operator for (1) is contracting in L^1.

The induction hypothesis holds at time $t_*=0$ because $X^{(1)}(0)=X^{(0)}(0)=X(0)$. We have thus reduced the proof of the theorem to demonstrating that if the induction hypothesis (18) holds at time t_*, then it must hold at time T_*. At time $T_*>t_*$, the time of the next interaction in w or among the free trajectories, two cases can occur.

Case # 1 *Two discontinuities, $\mathcal{S}_{(k,l)}$ and $\mathcal{S}_{(k,l+1)}$ from the free trajectories $F^{(k)}(t)$, meet at time T_*.* Let the trajectories (14) and (15) defined with respect to time T_* be distinguished by a superscript tilde from those defined with respect to time t_*. Then $\widetilde{X}^{(1)}\equiv X^{(2)}$ and Lemma 3 imply the induction hypothesis at time T_*,

$$
\begin{aligned}
d\big(\widetilde{X}^{(1)}(t),X(t)\big) &\le B(t)\cdot d\big(X^{(1)}(t),X(t)\big)\\
&\le B(t)A(t)\cdot d\big(X^{(0)}(t),X(t)\big), \qquad \forall t. \qquad (20)
\end{aligned}
$$

Case # 2 *Suppose the discontinuities $\mathcal{F}^{(k)}$ and $\mathcal{F}^{(k+1)}$ in w meet at time T_*.* A more general version of Lemma 4, presented in [8], would apply directly to this case. To keep the presentation short, we briefly explain how a slight re-interpretation of Lemma 4 would suffice to prove the induction hypothesis.

Suppose that the two discontinuities in w at time t_*, $\mathcal{F}^{(k)}$ and $\mathcal{F}^{(k+1)}$, interacted to form a discontinuity $\widetilde{\mathcal{F}}^{(k)}$. The trajectories $\widetilde{X}^{(1)}(T_*)=X^{(2)}(T_*)$ for all waves in $\mathcal{W}\setminus\widetilde{\mathcal{F}}^{(k)}$ and therefore it suffices to find bounds for the discrepancies involving only the waves in $\widetilde{\mathcal{F}}^{(k)}$. By definition, the trajectories $\widetilde{X}^{(1)}(T_*)|_{\widetilde{\mathcal{F}}^{(k)}}$ correspond to the positions $\widetilde{Y}(T_*)$ of waves in an entropy solution with $\widetilde{\mathcal{F}}^{(k)}$ as initial data, say $\widetilde{u}$. The trajectories $X^{(2)}(T_*)|_{\widetilde{\mathcal{F}}^{(k)}}$ correspond to the positions of waves in two entropy solutions with respectively $\mathcal{F}^{(k)}$ and $\mathcal{F}^{(k+1)}$ as initial data. In this case, $X^{(2)}(T_*)|_{\widetilde{\mathcal{F}}^{(k)}}$ is a trajectory of the form (15) for a subdivision $\mathcal{F}^{(k)}$, $\mathcal{F}^{(k+1)}$ of a total set of waves $\widetilde{\mathcal{F}}^{(k)}$. We now reinterpret Lemma 4 by replacing the positions $Y(t)$ of waves associated to a solution u having initial data $\mathcal{W}$, with the positions $\widetilde{Y}(t)$ of the waves in a solution $\widetilde{u}$ having as initial data $\widetilde{\mathcal{F}}^{(k)}$. Lemma 4 then provides a conservative matrix $C(t)$ such that for all $t\le\widetilde{t}_*=T_*$, we have

$$d\big(\widetilde{X}^{(1)}(t), X(t)\big)\big|_{\widetilde{\mathcal{F}}^{(k)}} \leq C(t) \cdot d\big(X^{(2)}(t), X(t)\big)\big|_{\widetilde{\mathcal{F}}^{(k)}}. \tag{21}$$

The definition of $C(t)$ can be extended in such a way that (21) holds for all $t > T_*$ by using Definition 3 and the fact that (14) implies $d/dt(\widetilde{X}^{(1)}) = d/dt(X^{(0)})$ for $t > T_*$.

Extending $C(t)$ trivially to all waves in $\mathcal{W}$ and combining this with Lemma 3 and the induction hypothesis at time t_*, we find

$$\begin{aligned} d\big(\widetilde{X}^{(1)}(t), X(t)\big) &\leq C(t) \cdot d\big(X^{(2)}(t), X(t)\big) \\ &\leq C(t)B(t) \cdot d\big(X^{(1)}(t), X(t)\big) \\ &\leq C(t)B(t)A(t) \cdot d\big(X^{(0)}(t), X(t)\big). \end{aligned}$$

Since $\widetilde{X}^{(0)} = X^{(0)}$, the induction hypothesis holds at time T^*.

3 Numerical Results

We present numerical experiments comparing the effectivity of (9) as an estimator of the true error in L^1. The "true" error was computed by comparing the approximate solution w to a numerical approximation of u, the exact solution with $u(\cdot, 0) = w(\cdot, 0)$. We used the 2nd order slope-limiter method of Goodman and LeVeque [3] to obtain a sufficiently precise approximation of u.

The first experiment considers initial data formed exclusively of shock waves. Given the values $[a_i]_{i=1}^N = [0.0, -0.25, -0.5, -2.5, -2.75, -3.0, -3.5]$, where $N = 7$, we define

$$w(x, 0) = \begin{cases} a_i & \text{if } x \in [(i-1), i], \\ a_1 & \text{if } x < x_0, \\ a_N & \text{if } x > x_N. \end{cases} \tag{22}$$

In Table 1, the error estimator (9) and the L^1 norm are presented, both evaluated at time $t = 8.0$. In Figure 1, we present the true error (solid) and the error estimator (dash) as a function of time for the experiment with $\Delta x = 0.125$. Experiments with other values of Δx confirmed the exactness of the error estimator. The temporary over-estimates of the error, as in Figure 1, are to be expected but beyond the scope of this paper.

We also present numerical experiments with initial data containing bock shock and rarefaction waves. Using the sequence of $N = 12$ values,

$$[a_i]_{i=1}^N = [0, -5, -3, -4, -1, -2, 3, 2, 4, 3, 5, 0], \tag{23}$$

we define initial data $w(\cdot, 0)$ in the same manner as (22). The solution converges to an N-wave profile and in Table 2 we compare the error and the error estimator at time $t = 50.0$, when $\|u\|_{L^\infty} \approx 1$. Figure 2 shows the comparison as a function of time for $\Delta x = 0.25$. This confirms numerically that the estimate (9) should continue to hold for arbitrary L^∞ initial data.

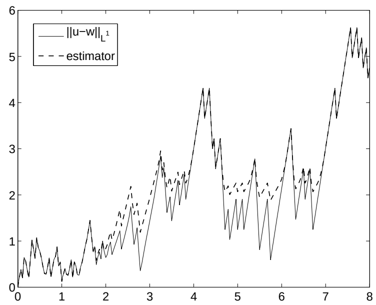

Fig. 1. Initial data (22)

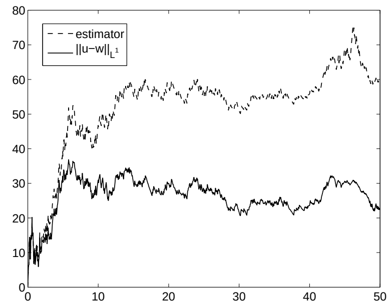

Fig. 2. Initial data (23)

Δx	$\|u-w\|_{L^1}$	$\sum_\alpha \|\sum_k R\|$
0.125	4.750	4.750
0.0625	0.0631	0.950
0.03125	5.969	5.969
0.015625	4.766	4.766

Table 1. Errors w.r.t. Δx for (22).

Δx	$\|u-w\|_{L^1}$	$\sum_\alpha \|\sum_k R\|$
0.5	49.576	93.465
0.25	29.903	57.716
0.125	10.340	30.559
0.0625	5.744	35.428
0.03125	2.466	24.739

Table 2. Errors w.r.t. Δx for (23).

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