# A Survey of Error Indicators and Error Estimators for Hyperbolic Problems

M. Laforest

M. A. Christon

T. E. Voth

Department of Mathematics, Colorado State University, Fort Collins, CO 80523-1874

Computational Physics R&D Department, Sandia National Laboratories, Albuquerque, New Mexico 87185-0819

Computational Physics R&D Department, Sandia National Laboratories, Albuquerque, New Mexico 87185-0819

Work performed under contract # A0343 with Sandia National Laboratory.

Abstract. We present a survey of error estimators for hyperbolic equations with special emphasis on conservation laws. Both empirical error indicators and rigorous error estimators are discussed. The indicators and estimators are examined in terms of their effectiveness for error estimation, the efficiency of the refinement they produce, and their computational cost.

# Contents

CHAPTER 1

# Introduction

Equations of hyperbolic type model physical systems whose global behavior is essentially controlled by the propagation and interaction of localized structures. During simulations, these structures also contribute the dominant part of the error and need to be adequately resolved. For these problems, the local refinement of the computational mesh is a natural method to minimize the computational cost for a fixed accuracy. In this paper we review the criteria used to estimate these local errors and to determine the regions requiring a refined grid.

The application of adaptive mesh refinement (AMR) to hyperbolic conservation laws presents some unique challenges. Solutions to hyperbolic conservation laws exhibit two properties which make them ideal candidates for AMR; local disturbances travel with finite speed and discontinuities occur naturally in nonlinear problems. The first feature implies that local refinement can be expected to improve the solution locally without adversely affecting the solution globally. The second feature would suggest that AMR would improve the resolution of discontinuities. On the other hand, the presence of discontinuities has created significant difficulties in the solution of these problems. Even the existence of solutions to generic systems of conservation laws in several space dimensions has not yet been established. As we will explain throughout this paper, it is in fact the simultaneous resolution of both smooth and discontinuous features of the flow that have led to difficulties. It is possible for computations to have well-resolved discontinuities, under-resolved smooth regions, and $\mathcal{O}(1)$ global errors [**52**].

These observations indicate that the error estimator controlling the process of AMR may have a significant impact on the quality of the solution. In theory, the error estimator should have a proven relation to the true approximation error. Unfortunately, many of the estimators we present only partially satisfy this condition. The search for error estimators with proven reliability is ongoing although there exists promising results for nonlinear and multi-dimensional hyperbolic problems [**1, 23, 25**]. On the other hand, many heuristic estimators, having little or no proven relationship to the true error, have been used with success in a broad range of problems. Following accepted conventions, we will henceforth use the terms *error estimator* and *error indicator* to distinguish between those quantities which rigorously bound the approximation error. and those for which no formal results exists.

In this paper, we have attempted to formulate a framework for the comparison of different error indicators. Four natural requirements for error indicators have emerged from our survey of the literature. Error indicators should (A) vanish under mesh refinement, (B) capture errors in both smooth and discontinuous regions, (C) capture both the cell and the transmitted errors, and (D) be computationally inexpensive. In the absence of numerical experiments specifically designed to compare error indicators, these four criteria have served as surrogate measures of assessment. We do note that Zhang et al. performed comparisons of a few error indicators for Euler's equations [**54**]. The significance of these conditions will be discussed in Chapter 3.

The error indicators we present are intended for the recent initiative to develop high-resolution algorithms for conservation laws within the ALEGRA shock-physics framework [**11**]. The goal is to implement schemes capable of modeling several materials with realistic thermodynamic models on block-structured adaptive grids. In this report we therefore focus our analysis to error indicators for either high-order finite difference or for finite-volume schemes, although most of the heuristic error indicators could also have been written for the finite element method. The last section is concerned with error estimators particular to the finite element method.

There are other important factors influencing the success of an adaptive algorithm but not directly related to either AMR or to hyperbolic conservation laws. Among these we mention the adaptive strategy used to implement the error indicator and the influence of the geometry of the grid on the solution. The first factor will be briefly examined in Section 3. The second factor is outside of the scope of this report and, in any case, it is hoped that a good adaptive algorithm can compensate for these types of errors. We refer to the survey of Berzins [**10**] for a discussion of the issues related to the geometry of the mesh.

We begin Chapter 2 with an overview of the numerical methods used in AMR and of the AMR algorithm of Berger et al. [**7, 8, 49, 50**]. The goal will be to present the context within which error indicators and estimators are used. This will also help to highlight some of the restrictions AMR places upon the model equation, the flow solver, and the associated computational expense. This section also contains a discussion of the adaptive strategies used to identify the cells needing refinement from the set of approximate values returned by the local error indicator or estimator.

In Chapter 3, three types of error indicators will be presented: feature indicators, reconstruction-based indicators, and residual-based indicators. None of these indicators have demonstrated relationships to the numerical error in computations of hyperbolic conservation laws. In many cases, particularly those based on Taylor's expansion, the presence of discontinuities within the solution casts serious doubts as to the relevance of these estimates. However, experience has shown that these indicators perform efficiently and quite accurately for many problems.

In Chapter 4, we present the rigorous error estimators for finite difference schemes of Gosse and Makridakis [**19**] and of Kröner and Ohlberger [**30**]. These two estimators bound the error but contain only coarse information on the source of the error. Section 2 briefly reviews the recent error estimators for finite element approximations of Johnson, Süli and their collaborators [**26, 47**]. Starting with linear hyperbolic problems, they have exploited Galerkin orthogonality and the stability of the dual problem to construct a wide range of $h$ and $hp$-adaptive algorithms. We will also address the computational challenges inherent in this new paradigm. These new estimators have also stimulated interest in the residual as an error estimator, as seen in the work of [**44, 45**]. Finally, in Chapter 5 we will conclude with some recommendations and some directions for future investigations.

CHAPTER 2

# Preliminaries

The purpose of this section is to introduce our notation for conservative finite difference and finite-volume methods. Although we have a broader scope of application in mind, they form a useful framework for the presentation of error estimators. For conservation laws, they are sufficient to illustrate most of the important conditions required of adaptivity. We also review Berger and Colella's adaptive algorithm and examine some adaptive strategies.

## 1. Numerical Methods

Our prototype for a hyperbolic problem will be a nonlinear conservation law in the form

$$u_t + \nabla_x \cdot f(u) = 0, \tag{1.1}$$

defined over some domain $(x,t) \in \Omega \times [0,T] \subset \mathbb{R}^{n+1}$ with prescribed boundary conditions everywhere along the boundary except at time $t = T$. The main assumption is that the Jacobian of the flux $Df(u)$ possess $d$ real and distinct eigenvalues for $u \in \mathbb{R}^d$. Solutions to these problems naturally develop discontinuities and an additional entropy-entropy flux pair must be specified in order to distinguish between the physical and physical solutions [**13**].

Approximate solutions to equation (1.1) are described throughout the grid by their cell averages. Suppose the region $\Omega \subset \mathbb{R}^n$ is partitioned into a grid $\{\kappa_i\}_{i\in\mathcal{T}}$ of convex polytopes satisfying

i) $\bigcup_i \kappa_i = \Omega$,
ii) $\kappa_i \bigcap \kappa_j \neq$ if and only if the intersection is a face of codimension 1 of both polytopes.

At a fixed time $t$, let $h(t) = \min_{i\in\mathcal{T}} \operatorname{diam}(\kappa_i)$. Suppose that the time interval $[0,T]$ has been discretized as $t_0 = 0 < t_1 < \ldots < t_N = T$. An approximate solution over this grid will be denoted by $u_h$ and have the cell averages

$$\frac{1}{|\kappa_i|}\int_{\kappa_i} u_h(x,t_m)\ dx \equiv u_i^m, \tag{1.2}$$

where $|\kappa_i|$ is the area of $\kappa_i$. If the approximation is piecewise constant then

$$u_h(\cdot, t_m)\Big|_{\kappa_i} = u_i^m.$$

We now describe a generic finite volume method. Let $s_{ij} \equiv \kappa_i \bigcap \kappa_j$ and $n_{ij}$ be the normal to the face $s_{ij}$ in the direction outwards from $\kappa_i$. Suppose that $F_{ij} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz and satisfies

i) $F_{ij}(u, u) = |s_{ij}| n_{ij} \cdot f(u)$ (consistency),
ii) $F_{ij}$ is increasing in the first variable and decreasing in the second (monotonicity).

The cell averages of the finite volume approximation $u_h$ are then defined to be

$$u_i^{n+1} = u_i^n - \frac{\Delta t_n}{|\kappa_i|} \sum_{j \in N_i} F_{ij}(u_i^n, u_j^n), \tag{1.3}$$

where $N_i$ is the set of all cells sharing a face with the $i$-th cell [**29**].

Definition (1.3) is a first order method but higher order methods have also recently been constructed [**5, 18**] by applying the flux-limiting techniques of MUSCL and ENO finite difference schemes. Briefly, a higher order approximation $\widetilde{u}_h$ is constructed by estimating the derivatives, possibly through a least-squares fit or interpolation. This new approximation is assumed to have the same cell averages as the original $u_h$. The flux evaluations in (1.3) are then replaced by quadratures over the cell boundaries. This reconstruction procedure can also be used to obtain error indicators, see Section 2.2.

In one space dimension, our notation can be simplified. In this case, the index of the partition $\mathcal{T}$ is a subset of the integers and space is partitioned by a sequence of points

$$x_{-p}, x_{-p+1}, \ldots, x_0, x_1, \ldots, x_{q-1}, x_q,$$

and the $i$-th cell is simply $\kappa_i = [x_{i-1}, x_i]$. If $|\kappa_i| = x_i - x_{i-1} \equiv h$ and $t_{n+1} - t_n \equiv k$ are independent of $i$ and $n$, then a conservative difference method can be written as

$$u_i^{n+1} = u_i^n - \frac{k}{h}\Big[F_i - F_{i-1}\Big], \tag{1.4}$$

where $F_i$ depends on the values of $u_h$ over a stencil of points centered at $x_i$. The flux need not satisfy condition ii).

## 2. Berger and Colella's AMR on structured grids

An approximate solution to (1.1) produced by the finite difference method (1.4) on a grid without a uniform mesh spacing cannot be conservative. This is easy to observe since the fluxes on neighboring cells, $k/|\kappa_i|F_i$ and $k/|\kappa_{i+1}|F_i$ do not cancel each other. The method of Berger and Colella [**8**] corrects this difficulty and, maybe more importantly, also incorporates first, independent timesteps for finer cells and secondly, an efficient error indicator based on Richardson extrapolation. These features warrant the special attention we attach to this algorithm.

In order to simplify our discussion of this version of AMR we assume that the spatial domain $\Omega$ is a rectangle in $\mathbb{R}^2$ with sides parallel to the coordinate axes. This allows us to ignore questions regarding the refinement of the grid near a curved boundary. Space is initially discretized by a family of nested *rectangular* grids covering the domain $\Omega$. The grids are nested in the sense that there exists a refinement ratio $r$ such that:

i) if the coarsest grid has constant mesh sizes $h_x$ and $h_y$ then a *level* $l$ fine grid has mesh sizes $h_x/r^l$ and $h_y/r^l$,
ii) every fine grid of level $l+1$ is contained inside a grid of level $l$ and begins and ends at points of the level $l$ grid,
iii) two adjacent cells are separated by at most one level of refinement.

These conditions will continue to hold as the grid is refined and/or coarsened. For a given rectangular grid of level $l$, the cell centers are denoted by $x_{i,j}$, and therefore satisfy $x_{i+1,j} - x_{i,j} = (h_x/r^l, 0)$ and $x_{i,j+1} - x_{i,j} = (0, h_y/r^l)$.

The numerical integrations proceed with the help of a conservative solver which can be written on the coarsest grid in two space dimension as

$$u_{i,j}^{n+1} = u_{i,j}^n - \frac{k}{h_x}\left[F_{i+1/2,j}^x - F_{i-1/2,j}^x\right] - \frac{k}{h_y}\left[F_{i,j+1/2}^y - F_{i,j-1/2}^y\right], \tag{2.1}$$

where $u_{i,j}^n$ is the average of $u$ over the $(i,j)$-th cell at time $t_n$, $k$ is the timestep, and $F_{i+1/2,j}^x$, $F_{i,j+1/2}^y$ are the numerical fluxes across respectively the left and upper cell boundaries.

To integrate the domain over a coarse timestep $k$, the grids are recursively integrated from the coarsest to the finest level. Assume that a coarse solution at level $l-1$ has been computed for times $t$ and $t + k/r^{l-1}$. From a level $l$ grid with known boundary data at time $t$, we begin by interpolating the boundary values for the timesteps $t + k/r^l, \ldots, t + (r-1)k/r^l$ using the boundary values from the coarse level $l-1$ grid. A number $r$ timesteps are then taken on the fine grid from time $t$ to time $t + k/r^{l-1}$.

The important steps are the modifications introduced to the level $l$ solutions by the finer grid solutions. If a level $l$ cell is covered by a refined level $l+1$ grid then after the integration of the finer grid, the value of the level $l$ cell is replaced by the average of the finer grid values. If a level $l$ cell, say indexed $(i,j)$, is adjacent to but not contained in, a level $l+1$ cell then, because of the first modification of the level $l$ solution, the value $u_{i,j}^{n+1}$ on the coarser grid needs to modified to maintain conservation conservation across the coarse/fine boundary. If the level $l$ cell $(i,j)$ is adjacent to level $l+1$ cells along the boundary between the coarse $(i,j)$ and $(i+1,j)$ cells, then the new value $\hat{u}_{i,j}^{n+1}$ over the coarse cell is given by

$$\hat{u}_{i,j}^{n+1} = u_{i,j}^{n+1} - \frac{k}{h_x}\Big[ - F_{i+1/2,j}^{\text{coarse}} + \frac{1}{r^2} \sum F_{i+1/2,j}^{\text{fine}} \Big], \tag{2.2}$$

where the sum occurs over the $r$ fine grid fluxes at the coarse/fine boundary and over the $r$ fine timesteps in $[t, t+\Delta t/r^l]$.

We consider the case where the error indicator is spatial, i.e. depends only on the solution computed at time $t$. Following Trangenstein [**51**], we choose a *regrid interval* $s \in \mathbb{N}$ and evaluate the error indicator for all grids of levels $l$ or finer if the current timestep $t/k^l$ is a multiple of the regrid interval. If a cell is flagged for refinement by the error indicator, then a buffer zone of cells within a distance $s$ of the original cell is also flagged for refinement. Moreover, all coarser cells containing the cell will also be refined. After the refinement, $s$ more timesteps can be taken without the 'error', or the disturbance, in the original cell propagating outside of the newly refined region. This observation is a consequence of the usual CFL timestep restriction which forces waves to travel through at most one cell per timestep.

When Richardson extrapolation is the error indicator, Trangenstein has proposed a few improvements over the implementation of Berger and Colella. These differences are discussed in detail in Section 2.1 but are summarized below. In his version of Richardson extrapolation, he was able to decrease the size of his buffer zones and the number of timesteps required for extrapolation. He also implemented timesteps on the finer grids that were not a multiple of the global refinement ratio and the coarse timesteps. This is particularly significant since adaptivity oftens uncovers increased wave speeds within the fine scale features of the flow. Although Richardson extrapolation might appear expensive in AMR, in fact the majority of cells belong to the finest level where the error is never computed.

We conclude with a few remarks. First of all, the key to the reconstruction of the coarse grid solutions according to (2.2) is clearly the conservative form of the scheme (2.1). In particular, using this algorithm for viscous flows is still an open problem [**12**]. Second, several extensions to this algorithm already exist. Berger and LeVeque have adapted LeVeque's second-order wave-propagation scheme to AMR and thus extended AMR to handle equations not in conservation form, with source terms, and logically rectangular curvilinear grids [**9**]. Unfortunately, other solvers have yet to handle these complications within the context of AMR. The cartesian grid can also be adapted to a piecewise linear boundary, at the cost of introducing cells cut by one or more planes. The boundary conditions and the numerical schemes can be adapted to this situation [**3**]. Trangenstein [**50**] has implemented AMR for problems in solid mechanics. His method permitted adaptive refinement of the equation of state even when it involved a time history. Aftomis [**2**] has proposed a method for upwinding on hexahedral meshes. His grids physically resemble the cartesian grids of Berger and Colella [**8**] but in his method, values are node-centered and a dual mesh is required. In particular, this allows him to model viscous forces.

## 3. Adaptive Strategies

It has been remarked [**52**] that the final adapted solution is also sensitive to the choice of an adaptive strategy. By adaptive strategy we mean the algorithm through which the numerical values of the error indicators are used to identify a set of cells needing refinement or unrefinement. Here, we review a few possible adaptive strategies and make some recommendations.

The first natural adaptive strategy is to refine cell $i$ when its error satisfies

$$\frac{e_i}{e_{\max}} > \alpha_{\max},$$

and to unrefine the cell when

$$\frac{e_i}{e_{\max}} < \alpha_{\min},$$

where $e_{\max} = \max e_j$ and $\alpha_{\max}$, $\alpha_{\min}$ are user-specified constants. One of the drawbacks of this strategy is that, as seen in the experiments of Warren and his collaborators, the final adapted mesh can be sensitive to the choice of $e_{\max}$ and $\alpha_{\max}$. These difficulties are probably related to the fact that this strategy does not distribute the error uniformly over the grid.

A second strategy which explicitly attempts to improve the distribution of the error has been proposed by Kallinderis [**27**]. If $\overline{e}$ is the average of the error $e_i$ over the mesh and $\sigma$ is its standard deviation then we can flag for refinement (unrefinement) any cell $i$ for which

$$e_i - \overline{e} > m + \lambda\sigma \qquad ( e_i - \overline{e} < m - \lambda\sigma ),$$

where $\lambda$ is a positive user-specified constant.

A third possibility is to refine (unrefine) a fixed fraction of the mesh during each adaptation cycle. This can be accomplished by constructing a histogram of the errors which also contains pointers to the cells. After having computed the error of every cell, it is then straightforward to identify the fraction needing adaptivity. When the error distribution of the cells is normal, this method is equivalent to the previous strategy.

Although each of these strategies involve parameters whose effect on the final adaptive solution is difficult to assess, the last two strategies are expected to produce a more reasonable distribution of the error.

CHAPTER 3

# Error Indicators

In this chapter we review the commonly used error indicators and assess their relative strengths and weaknesses. The error indicators in this chapter can be roughly categorized as either feature indicators, reconstruction-based indicators, or residual-based indicators. Feature indicators identify variations and discontinuities within the approximate solution and assume that these are the dominant source of error. Reconstruction-based indicators compare a coarse versus a fine approximation to extrapolate a measure of the error. Residual-based indicators are derived from the local residual. Throughout this section, $\delta_i$ will denote the value of an error indicator on the $i$-th cell. This value will be called an *error estimate* and it may or may not have any relationship to the *true error*, which is the difference between the exact and approximate solution over the cell.

Before discussing particular error indicators, it is possible to list some simple requirements that an error indicator should satisfy. First of all, an error indicator should behave like the true error in the sense that $\delta_i \to 0$ as the mesh size $h_i$ vanishes. Error indicators satisfying $\delta_i = \mathcal{O}(1/h_i^r)$ can lead to pathological behavior, as seen in examples of Dannenhoffer [**14**]. In fact it is easy to see that such an error indicator would lead to excessive refinement. Clearly the correct solution cannot be obtained unless the errors, and therefore the mesh sizes, are reduced everywhere. In the same vein, the regions not initially flagged for refinement by an $\mathcal{O}(1/h_i^r)$ indicator will remain unrefined, thus possibly omitting some significant errors. Warren [**52**] has exhibited an example of a 2-D flow over an airfoil where the lack of refinement in smooth regions led to a solution with a well-resolved shock, under-resolved smooth regions, and $\mathcal{O}(1)$ errors. Although this particular example occurred with an $\mathcal{O}(1)$ indicator, this type of error could have also occurred for $\delta_i = \mathcal{O}(1/h_i^r)$.

Our second requirement is concerned with the presence of discontinuities within approximate solutions to hyperbolic problems, especially for problems in conservation form. One must recognize that for hyperbolic conservation laws significant errors are generated in both the smooth and discontinuous regions of the flow. The error indicator should have the ability to identify errors in both of these regions. This is a difficult requirement to verify in practice since shock discontinuities generate $\mathcal{O}(h_i)$ errors while smooth regions generate $\mathcal{O}(h_i^p)$ errors in discontinuous regions, where $p$ is the order of the scheme.

Our third requirement is based on the concept of cell and transmitted error [**21**]. Let $u$ denote the exact solution to (1.1) and let $u_h$ denote an approximate solution. On the $i$-th cell $\kappa_i$, let $\widetilde{u}_h$ be the exact solution to the problem

$$\begin{aligned} \partial_t \widetilde{u}_h + \nabla_x \cdot f(\widetilde{u}_h) &= 0, \\ \widetilde{u}_h\big|_{\partial\kappa_i} &= u_h\big|_{\partial\kappa_i}. \end{aligned} \tag{0.1}$$

On the $i$-th cell, we then define the *cell error*,

$$e_i^{\text{cell}} \equiv \widetilde{u}_h - u_h, \tag{0.2}$$

and the *transmitted error*,

$$e_i^{\text{trans}} \equiv \widetilde{u}_h - u. \tag{0.3}$$

These two errors have been constructed to satisfy the relation

$$e_i = (u - u_h)\big|_{\kappa_i} = e_i^{\text{cell}} + e_i^{\text{trans}}. \tag{0.4}$$

This formulation emphasizes that the cell error vanishes when the conservation law (0.1) over the $i$-th cell can be solved exactly. The transmitted error however would vanish only if the conservation law is solved exactly outside of the cell.

It is easy to see that the cell error could have also been defined as the solution to the *error equation*

$$\begin{aligned} \partial_t e_i^{\text{cell}} + \nabla_x \cdot \widetilde{f}(e_i^{\text{cell}}) &= r_h, \\ e_i^{\text{cell}}\big|_{\partial\kappa_i} &= 0. \end{aligned} \tag{0.5}$$

The error equation and its modified flux $\widetilde{f}$ will be discussed later in Section 3.3. With this definition of the cell error, the transmitted error could have been chosen to satisfy relation (0.4).

Houston and Süli [**23**] have studied numerically the relationship between the cell error, the transmitted error, and the true error. They observed that the true error was well correlated to the transmitted error and that the cell error was well correlated to the size of the residual. Unfortunately the local cell error correlates poorly to the true error. This was to be expected for hyperbolic problems where information, and hence errors, are propagated along characteristics. The transmitted error at a cell can be bounded by the cell errors produced in the domain of dependence of that cell [**47**]. This situation can be roughly summarized as follows; the *source* of errors are the cell errors but the *observed* errors are the transmitted errors.

Any error indicator will therefore be faced with these properties of the error. If the purpose of the estimator is to estimate the true error than it should identify the transmitted error. If the purpose of the estimator is to identify the regions where errors are generated, then it can be satisfied to identify the cell error. Unfortunately the relationship between the observed error $e_i$ and the strength of the error source $e_i^{\text{cell}}$ is nonlinear. Therefore for any fixed global threshold, local refinement based solely on the cell error may not improve the quality of the approximation.

For these reasons, we propose as a third requirement that an error indicator should be sensitive to the transmitted error first or, when this is not possible, to the cell error. In our comparisons, it will be important to identify the components of the error measured by the error indicator.

Our fourth requirement is that the error indicator be inexpensive. An obvious upper bound on the cost is that the implementation of the error indicator and the subsequent mesh refinement should be lower than the cost of simply doubling the number of grid points and computing a new approximation. For a rigorous and reliable error estimator, the threshold for an acceptable cost may be higher.

Our analysis of error indicators will therefore be based on the four criteria above, namely,

A) the indicator $\delta_i$ should vanish as $h_i \to 0$,
B) the indicator should measure the errors in both the smooth and discontinuous regions of the flow,
C) the indicator should measure the transmitted error, or at least the cell error,
D) the indicator should be inexpensive.

## 1. Feature Indicators

Feature indicators attempt to identify the location and the strength of either discontinuities or variations in the smooth flow. The underlying principle is that such local features of the solution are the dominant source of errors in computations. To some extent, this statement is correct but in practice it can often be an oversimplification [**15**].

The main characteristics of shock indicators are the flow variables on which they depend and their order with respect to the mesh size. In general, feature indicators measure only the change in density but many have also argued for pressure[**2**] or entropy [**46**]. The jump in

| Name | expression | $L^\infty$ | $L^2$ | $L^1$ | $H^{-1}$ |
|---|---|---|---|---|---|
| 1st divided difference | $\frac{\partial v}{\partial x}$, or $\frac{1}{v}\frac{\partial v}{\partial x}$ | $\mathcal{O}(\frac{1}{h})$ | $\mathcal{O}(\frac{1}{\sqrt{h}})$ | $\mathcal{O}(1)$ | $\mathcal{O}(\sqrt{h})$ |
| 1st undivided difference | $\Delta v \approx h\frac{\partial v}{\partial x}$, or $\frac{1}{v}\Delta v$ | $\mathcal{O}(1)$ | $\mathcal{O}(\sqrt{h})$ | $\mathcal{O}(h)$ | $\mathcal{O}(h^{3/2})$ |
| 2nd divided difference | $\frac{\partial^2 v}{\partial x^2}$, or $\frac{1}{v}\frac{\partial^2 v}{\partial x^2}$ | $\mathcal{O}(\frac{1}{h^2})$ | $\mathcal{O}(\frac{1}{h})$ | $\mathcal{O}(\frac{1}{h^{3/2}})$ | $\mathcal{O}(\frac{1}{\sqrt{h}})$ |
| 2nd undivided difference | $\Delta^2 v \approx h^2\frac{\partial^2 v}{\partial x^2}$, or $\frac{1}{v}\Delta^2 v$ | $\mathcal{O}(1)$ | $\mathcal{O}(h)$ | $\mathcal{O}(\sqrt{h})$ | $\mathcal{O}(h^{3/2})$ |

TABLE 1. Common error indicators and their orders near a discontinuity.

entropy is not sensitive as the gradient of density and pressure [**44**], even though there is theoretical evidence that the closely related entropy production might be a sensitive indicator [**32, 40**]. Aftomis [**2**] has also used the velocity magnitude as an indicator of smooth features of the flow. For gas flow, density is usually sufficient since it is the only variable which is discontinuous across both shock and contact discontinuities. Experience therefore suggests density, pressure, or velocity magnitude are useful quantities but the final choice should be problem dependent. In any case, we will assume that $v$ denotes some function of the state variables and express the feature indicator in terms of this generic variable.

In Tables 1 and 2, we have presented some of the commonly used feature detectors. The norms are computed in one space dimension. For each extra space dimension, the order of these norms increases by one power of the mesh size. The weak Sobolev norm appearing in these tables is discussed later in section 3. As we have argued earlier, an error indicator which vanishes as $h \to 0$ is to be preferred.

The 1st and 2nd divided differences target for refinement different regions in the neighborhood of a discontinuity. The 1st divided difference attains its maximum at the center of a discontinuity. This indicator therefore leads to unnecessary refinement in the center of discontinuities. On the other hand, the 2nd divided difference reaches its maximum at the edges of a discontinuity. It therefore helps to resolve the important regions around a shock but may not help to sharpen the shock, in those cases where it was initially spread over too large of a region [**39**].

| Name | expression | $L^\infty$ | $L^2$ | $L^1$ | $H^{-1}$ | references |
|---|---|---|---|---|---|---|
| 1st divided difference | $\frac{\partial v}{\partial x}$ | $\mathcal{O}(1)$ | $\mathcal{O}(\sqrt{h})$ | $\mathcal{O}(h)$ | $\mathcal{O}(h^{3/2})$ | [**41, 53**] |
| 1st undivided difference | $h\frac{\partial v}{\partial x}$ | $\mathcal{O}(h)$ | $\mathcal{O}(h^{3/2})$ | $\mathcal{O}(h^2)$ | $\mathcal{O}(h^{5/2})$ | [**2, 35, 52**] |
| 2nd divided difference | $\frac{\partial^2 v}{\partial x^2}$ | $\mathcal{O}(1)$ | $\mathcal{O}(\sqrt{h})$ | $\mathcal{O}(h)$ | $\mathcal{O}(h^{3/2})$ | |
| 2nd undivided difference | $h^2\frac{\partial^2 v}{\partial x^2}$ | $\mathcal{O}(h^2)$ | $\mathcal{O}(h^{5/2})$ | $\mathcal{O}(h^3)$ | $\mathcal{O}(h^{7/2})$ | [**2, 38**] |

TABLE 2. Common error indicators in a smooth region.

Our list of error indicator is exhaustive in the sense that any remaining feature indicators are equivalent, up to some normalization, to indicators in the tables above. Among some of the other alternatives we mention the indicators of Löhner [**35**],

$$\frac{|u_{j+1} - 2u_j + u_{j-1}|}{|u_{j+1} - u_j| + |u_j - u_{j-1}| + C}, \tag{1.1}$$

here written in 1-D centered at the $j$-cell. The term $C$ is introduced to filter out small oscillations. The parameter $C$ can be taken to be either a multiple of the local average of $u_h$ or

$$C = \alpha\Big(|u_{j+1} - u_j| + |u_j - u_{j-1}|\Big),$$

for some constant $\alpha$. This indicator has the same characteristics as the normalized 2nd divided difference, but is bounded above by unity.

In practice, it is useful to couple several of these error indicators. The 2nd divided difference of pressure might be used to detect sharp discontinuities and the 1st divided difference of density might be used to locate the smooth regions requiring refinement. This combination of indicators was reported to give good results on demanding problems [**2**]. The second divided difference of velocity was also used to detect regions of shear flow.

Despite their popularity, these indicators possess little theoretical support and years of numerical experimentation have provided almost no practical guidelines for their reliable use. With respect to condition A), Tables 1 and 2 indicate that in 1-D, only the 1st derivative in the $H^{-1}$ norm is appropriate, while in 2-D or 3-D the 2nd derivative can also be used as an error indicator provided it is adequately weighted. In smooth regions, such indicators are insensitive to flow features like rarefaction waves or to the identification of sonic points (where some 2nd order schemes lose an order of accuracy). These indicators therefore fail to satisfy condition B). Feature indicators provide no information other than accidental, on the cell error or the transmitted error. It is regrettable that there are no systematic quantitative studies of the performance of feature indicators. Without these, it is difficult to make blanket statements on their characteristics.

In conclusion, this family of error indicators is susceptible to a wide array of pathological behavior [**52**]. Moreover, since approximate solutions to hyperbolic conservation laws can degenerate under the influence of AMR, these types of error indicators can confuse the analysis of the final solution. The adaptive strategy should therefore depend at most marginally on such error indicators. The potential troubles caused by such indicators outweigh their main advantage, that is their low cost.

## 2. Reconstruction-based Indicators

The first step in the evaluation of this class of indicators is the construction of an improved approximate solution. The true error is then assumed to be proportional to the difference between the original and the improved approximation. These methods provide reliable error estimates, at least for smooth solutions, the improved solution may be prohibitively high since it involves computing a second more accurate solution. In special circumstances, the additional cost of the second solution can be offset. We will discuss two reconstruction-based indicators: Richardson extrapolation and local reconstructions.

**2.1. Richardson Extrapolation.** Richardson extrapolation is a popular error estimation procedure for structured grids. Assuming the solution is sufficiently smooth, a coarse and a fine grid solution are used to extrapolate the leading order term of the error. Despite the fact that the smoothness assumption is not satisfied for hyperbolic conservation laws, the estimate delivers surprisingly good results. After a general discussion of Richardson extrapolation, we will present some of the most important aspects of economically implementing this error indicator into an algorithm for AMR.

The first step in Richardson extrapolation is to obtain an estimate of the order of convergence $p$ of the scheme, as defined by the relation

$$v - u_h \approx \mathcal{O}(h^p). \tag{2.1}$$

The order can be computed either prior to the final computation, through grid convergence studies or, concurrently with the final calculations. In practice though, the observed order of convergence is problem dependent. This is due to the presence of discontinuities, the use of nonlinear limiters in high order schemes [**16, 43**]. Generally speaking, a flow without discontinuities will exhibit an order of convergence equal to one more than the formal order of the scheme.

Let $v$ be the exact solution and let $u_h$ be an approximate solution over a grid of mesh size $h$. Assume that the order of convergence $p$ in (2.1) is known. Recall that $k$ is the timestep subject to the constraint that $k/h$ is constant. To compute the error indicator, assume that $u_h$ and $v$ coincide at time $t$ and take $r$ fine timesteps for a cumulative error of

$$u_h(\cdot, t+rk) - v(\cdot, t+rk) \approx Crk^p. \tag{2.2}$$

Now coarsen the approximate solution $u_h$ by a factor $r$ and take one coarse timestep $rk$ to construct an approximation satisfying

$$u_{rh}(\cdot, t+rk) - v(\cdot, t+rk) = Cr^p k^p. \tag{2.3}$$

To solve for the error over one fine timestep, asymptotically $Ck^p$, use (2.2) and (2.3) to write

$$\begin{aligned} u_h(\cdot, t+k) - v(\cdot, t+k) \approx Ck^p =& \frac{C(r^p - r)k^p}{r^p - r} \\ =& \frac{u_{rh}(\cdot, rk) - u_h(\cdot, rk)}{r^p - r}. \end{aligned} \tag{2.4}$$

Therefore, in Richardson extrapolation the error indicator in the norm of (2.1) over the $i$-cell for one fine timestep is

$$\delta_i^{\mathrm{R}} = \frac{||u_{rh} - u_h||_{\kappa_i}}{r^p - r}. \tag{2.5}$$

If the order of convergence was not determined a priori then it could have been determined during AMR. Using three approximate solutions $u_h$, $u_{rh}$, and $u_{r^2h}$ from three successive levels of refinement then the order of convergence could have been estimated by

$$p = \ln\Big(\frac{||u_{r^2h} - u_{rh}||}{||u_{rh} - u_h||}\Big) \Big/ \ln(r). \tag{2.6}$$

The error indicator could then have been computed from the two finest approximations using (2.5). We remark that $p$ can be made into a pointwise function if the norm in (2.6) is replaced by pointwise values. Roy [**43**] has related the local variation in the observed order of convergence $p$ to the propagation of $\mathcal{O}(h)$ errors from shock fronts into smooth regions. In this case, a conservative estimate on $p$ will overpredict the true error while an overestimate of $p$ will underestimate the true error.

We now turn specifically to the implementation of Richardson extrapolation inside Berger and Colella's AMR framework [**8**] In AMR, there is a fixed integer $s$ called the regrid interval such that at each $s$ timesteps on a fixed level, the algorithm must determine the cells needing refinement. If this is such a timestep for a grid of level $l$ then before the next timestep is taken, the error over all subgrids of level $m \leq l$ must be computed. On a level $m$ grid, one takes one fine timestep forward, say of size $k$, and compares this with the result of one coarse timestep $sk$ of a coarsened solution from the previous $(s-1)$st timestep of the level $m$ grid. This is a more efficient strategy [**51**] then the method of Berger and Colella, where they take $s$ fine timesteps and compare the result to an approximation obtained from one coarse timestep. Another difference between these two approaches is that the regrid interval $s$ can be distinct from the refinement ratio $r$. Notice that the errors need not be computed on the finest grid since this level will never be refined. This last observation allows a cost efficient implementation of Richardson extrapolation. We also note that Richardson extrapolation always flags for refinement regions containing discontinuities. Most users [**7**] of this indicator have therefore found it convenient to begin error estimation with a feature indicator and then to apply Richardson extrapolation on the remaining unflagged regions.

Although the assumption of smoothness implicit in (2.1) may not always hold, many numerical studies have shown that this indicator is highly reliable [**24, 42**]. This indicator satisfies condition A) and can detect changes in the order of convergence when $p$ is computed locally [**43**]. Transmitted errors can also be observed when the transmitted error is of a different order than the underlying cell error [**43**]. As we have pointed out, the cost of this indicator can often be significantly reduced through careful planning. Sometimes this might even involve turning off the indicator in specific regions [**7**]. For example in Berger and Colella original implementation with three levels of refinement the integrations required for Richardson extrapolation represented only 3% of the total time spent on all integrations steps.

For the reasons mentioned above, we believe that Richardson extrapolation represents a benchmark for other indicators on structured grids. It should be noted though that this indicator estimates only the error in the variables $u$ and therefore cannot offer information on the error of some functional of the solution. An approximation with a small global error might not correspond to the engineering goals for the accurate approximation of a functional of the solution, like drag or lift.

**2.2. Local Reconstruction.** Another attractive method for the construction of improved approximations is to estimate the higher order derivatives of the solution. This might involve computing divided differences or solving a least-square problem. In all of these cases, the improved estimate will depend only on nearby values of the coarse approximation.

The first method to accomplish this uses high order reconstructions of the type found in ENO-like schemes [**5, 18**]. Given an initial approximation for the cell averages $u_h$, we want to construct a piecewise-linear approximation $\widetilde{u}_h$. The reconstruction algorithm should preserve cell averages,

$$u_i^n = \frac{1}{|\kappa_i|} \int_{\kappa_i} \widetilde{u}_h(x, t_n)\ dx, \tag{2.7}$$

and should reconstruct linear solutions exactly. The simplest algorithm computes a vertex-based average of the divided differences of the neighboring cells. Another possibility is to use a least-square approximation of the derivatives. Once a first approximation is obtained, one can also use a slope-limiter to enforce monotonicity in the reconstruction [**5**]. The error estimator would then be

$$\zeta_i^{\text{rec}} = \int_{\kappa_i} |\widetilde{u}_h - u_h|\ dx. \tag{2.8}$$

A second technique, popular in the finite element community, is the Zienkiewicz-Zhu estimator [**57**]. The goal is to find an improved approximation to the *spatial gradient* of the solution, denoted $g(u_h)$. Assuming such a $g(u_h)$ could be found then

$$\zeta_i^{\text{ZZ}} = ||g(u_h) - \nabla_x \cdot u_h||_{\kappa_i}. \tag{2.9}$$

Notice that the integrand is similar to the spatial term in the momentum equations for Euler equations. It is therefore tempting to see the relationship of the Zienkiewicz-Zhu estimator to a residual-based estimator.

In several studies [**24, 54**], indicators based on local reconstructions performed poorly, especially for supersonic flows and flows with discontinuities. These estimators often missed important regions of error and even global patterns in the error. This appears to be a consequence of the fact that the such local error indicator do not detect transmitted errors. The situation seems worse for the Zienkiewicz-Zhu estimator. In one example of the Zienkiewicz-Zhu estimator applied to a compressible flow [**39**], a shock wave reflected off a boundary and was spread over several cells but the estimator prevented refinement in the center of the wave. The trouble in this case is the estimators $\mathcal{O}(1)$ behavior in the center of the shock.

A review of the literature therefore reveals the unexpected fact that reconstruction-based indicators grossly fail for hyperbolic problems. Although they are frequently used on elliptic and parabolic problems, the particular nature of hyperbolic problems is not captured by these indicators. The local reconstruction of a higher-order approximation appears to be less reliable than Richardson extrapolation.

## 3. Residual-based Indicators

Following the example of a posteriori error estimation for elliptic and parabolic problems using finite elements, the residual has been studied as a potential error indicator. Recently, the work of Johnson, Süli [**21, 22, 26**] and their collaborators has demonstrated that the residual in the $H^{-1}$ norm can measure both the cell and the transmitted error in the $L^2$ norm for linear hyperbolic problems. In this section, we examine two types of error indicators derived from the residual. The first is essentially the residual measured in some norm. The second is the solution to the error equation where the residual appears as a source term. For a theoretical discussion supporting the residual as an error indicator, we refer to Section 2.

**3.1. The residual.** There have been studies examining the residual $r_h$ in the following norms: $||r_h||_{L^1}$ [**54**], $||r_h||_{L^2}$, $h||r_h||_{L^2}$, $||r_h||_{H^{-1}}$ [**44, 54**]. The residuals in the $L^1$ and $L^2$ norms sometimes appear as reasonable error indicators [**54**] but in 1-D and 2-D they fail to vanish as $h \to 0$ and therefore will not be discussed here. Instead, we will concentrate on $h||r_h||_{L^2}$ or $||r_h||_{H^{-1}}$ for which there is theoretical support. We note that Süli and his collaborators [**21**] have shown that these norms are equivalent over discrete grids.

In the case of the weak Sobolev norm $H^{-1}(\kappa)$ over an element $\kappa$,

$$||r_h||_{H^{-1}(\kappa)} = \sup_{\phi \in H_0^1(\kappa)} \frac{\left| \int_\kappa r_h \phi \; dx \right|}{||\phi||_{H^1}}, \tag{3.1}$$

the supremum over the space $H_0^1(\kappa)$ is usually approximated by the maximum over some finite-dimensional subspace of $V \subset H_0^1(\kappa) = \{u \big| \, u, \partial_x u \in L^2, u|_{\partial\kappa} = 0\}$ [**44, 54**]. This subspace can be created by introducing a subgrid over $\kappa$ and taking

$$||r_h||_{H^{-1}(\kappa)} \approx \max_{\phi \in V} \frac{\left| \int_\kappa r_h \phi \; dx \right|}{||\phi||_{H^1}}. \tag{3.2}$$

The $H^{-1}$ norm of the residual in essence measures the total variation of the solution. The following calculation roughly demonstrates this. Observe that if $w$ is a smooth function with compact support then

$$
\begin{aligned}
|w|_{\mathrm{TV}(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |\nabla \cdot w| \; dx \\
&= \sup_{\substack{\phi \in C_0^\infty(\mathbb{R}^d) \\ \sup|\phi|=1}} \int_{\mathbb{R}^d} \phi \nabla \cdot w \; dx \\
&= \sup_{\substack{\phi \in C_0^\infty(\mathbb{R}^d) \\ \sup|\phi|=1}} \int_{\mathbb{R}^d} w \nabla \cdot \phi \; dx \\
&= \sup_{\phi \in C_0^\infty(\mathbb{R}^d)} \frac{\left| \int_{\mathbb{R}^d} w \nabla \cdot \phi \; dx \right|}{||\phi||_{L^\infty}} \qquad (3.3)
\end{aligned}
$$

Rewriting the residual in (3.1) as

$$
r_h = \partial_t u_h + \nabla_x \cdot f(u_h) = \nabla_{(t,x)} \cdot \big(u_h, f(u_h)\big),
$$

then the analogy between definitions (3.1) and (3.3) suggests that $||r_h||_{H^{-1}}$ measures the total variation in space and time of $\big(u_h, F(u_h)\big)$. The similarity between the test spaces $H_0^1$ and $C_0^\infty$ in each of these definitions supports such a comparison. We note that when $u_h$ is piecewise constant $||r_h||_{H^{-1}}$ is equivalent to measuring the strengths of jumps across cell interfaces.

Sonar [**44**] observed that the residual indicators $h||r_h||_{L^2}$ and $||r_h||_{H^{-1}}$ are at least as discriminating as most feature indicators. They vanish as $h \to 0$ and Houston and Süli remarked that the cell error is well correlated to the residual [**23**]. Both Sonar and Zhang have noticed that this error indicator is noisy, even in smooth regions, and may lead to unnecessary refinement. In an important comparison between Richardson extrapolation, local reconstruction, the error equation, and the residual [**54**], it was found that only the residual in the $L^1$ norm flagged sonic points for refinement. In this sense, the residual does capture some information on the order of accuracy of the scheme, although this property disappeared when the residual was measured in the $L^2$ norm. The residual is also rather inexpensive to compute. In conclusion, the residual is to be preferred over a feature indicator but it cannot be used as a global error indicator since it cannot measure transmitted errors.

**3.2. A second-order residual.** In recent work, Thomas and Sonar [**48**] have proposed replacing the residual of the original approximation

$u_h$ by the residual of a smoothed approximation $\widetilde{u}_h$. An equivalent formulation using a weak form of the conservation law has been studied by Karni, Kurganov, and Petrova [**28**]. Suppose that we are given a triangular mesh together with its dual mesh. If a piecewise constant solution $u_h$ is given on the first mesh then by linear interpolation, a piecewise linear approximation $\widetilde{u}_h$ can be constructed on the dual mesh. By a direct calculation of the flux function, a second-order accurate expansion for the steady-state residual is found

$$\nabla \cdot f(\widetilde{u}_h) = \sum_{|\alpha| \leq 1} a_\alpha h^\alpha + \mathcal{O}\big(||h||^2\big),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $h^\alpha = h_1^{\alpha_1} \cdots h_d^{\alpha_n}$, and the $a_\alpha$ are combinations of the derivatives of $f$. The error indicator is then

$$\big|\big|\nabla \cdot f(\widetilde{u}_h)\big|\big|_{L^2} = \sum_{|\beta| \leq 2} b_\beta I_\beta,$$

where the $I_\beta$ depend only on the geometry of the mesh. The $L^2$ norm was chosen for the sake of computational expediency. We mention that the terms $I_\beta$ are $\mathcal{O}(h^{|\beta|+1})$ and therefore vanish with decreasing grid size. An implementation of this error indicator over structured meshes is discussed in [**33**].

This indicator was compared with the residual in the $H^{-1}$ norm for steady and unsteady flows [**48**]. Their indicator gave better predictions of the error than the residual in $H^{-1}$ ($10^{-3}$ vs $10^{-2}$ for an actual error of $10^{-6}$). The important flow features were better resolved with the smoothed residual but most importantly, the smooth residual was not as noisy as the $H^{-1}$ indicator. This error indicator was tested on a transonic flow problem over an NACA 0012 airfoil and on the forward-facing step problem of Woodward and Colella. Although this indicator could be extended beyond piecewise constant approximations $u_h$, the authors observed instabilities in their computations with higher-order approximations $\widetilde{u}_h$. This smoothed residual therefore provides essentially the same information as the residual from the previous section but with increased sensitivity in the smoother regions of the flow.

**3.3. The error equation.** The error in a conservation law satisfies a conservation law with a modified flux and the residual as a source term. Let $v$ and $u_h$ be, respectively, exact and approximate solutions to equation (1.1). Define the residual $r_h$ of the approximation $u_h$ to be

$$\partial_t u_h + \nabla_x \cdot f(u_h) \equiv r_h. \tag{3.4}$$

The residual $r_h$ is computable a posteriori and is zero precisely when $u_h$ is an exact solution. Subtracting (1.1) from (3.4) and writing $e = u_h - v$, we have the following equation for the error,

$$\partial_t e + \nabla_x \cdot \widetilde{f}(e) = r_h, \tag{3.5}$$

where

$$\widetilde{f}(e) = \int_0^1 \nabla \cdot f\big(\theta u_h + (1-\theta)(u_h - e)\big)\ d\theta.$$

In this context, a stability estimate for the non-homogeneous equation (3.5) is equivalent to an a posteriori error estimate for the approximation $u_h$.

In [**55**], the equation (3.5) was solved to determine the global error. They remarked that the equation had to be solved with a method of at least one order higher than the method used to compute the approximation $u_h$. Moreover, the second problem (3.5) was at least as demanding as the original (1.1). Although in theory, these error estimators satisfy conditions B) and C), in practice they are not as sensitive to flow features as Richardson extrapolation [**24**]. This approach to error estimation is therefore not recommended.

CHAPTER 4

# Error Estimators

A posteriori error estimators are rigorous quantities which bound the difference between the exact and approximate solutions. Such estimators result from an analysis of the stability properties of the underlying equation. Unfortunately, the stability results for hyperbolic systems of nonlinear conservation laws are not available for most numerical methods [**31, 34, 36**].

A first approach extended these stability results into rigorous error estimates for finite difference and finite volume methods [**19, 30, 32**]. In practice, these estimates for nonlinear have been crude since they often reduce to the local variation of the solution, i.e. first divided differences. These rigorous estimators have indicated that entropy production along discontinuities is a measure of the error generated by discontinuities. A brief sample of these estimates will be given in Section 1.

A second approach has been developed from the stability theory of Friedrichs for linear systems of hyperbolic type [**17**]. The error estimator is constructed as a weighted combination of the residuals and with a weight obtained by solving an associated dual problem [**23, 26**]. Although formally, the existence and stability of the solution to the dual problem are still open questions, these issues can be resolved by introducing viscosity to the original and the dual problem. These error estimators have the important property that they can be also be used to measure the error in a functional of the solution. This makes the method attractive for practical applications. Duality techniques will be the subject of Section 2.

## 1. Estimators for Finite Difference and Finite Volume Methods

In this section, we briefly examine two rigorous error estimators developed for monotone finite difference and finite volume methods. The restriction on monotonicity implies that these estimators apply only to a class of first-order schemes which satisfy discrete entropy inequalities. Moreover, these error estimators measure the total variation of

$u$ and its flux $f(u)$, as in Section 3.1, and in this sense are similar to gradient-based indicators. These two characteristics therefore make them of limited practical use.

We begin by describing the error estimator of Gosse and Makridakis [**19**] for scalar conservation laws in one space dimension. Let $v$ and $u$ denote respectively the exact and approximate solutions to the conservation law (1.1). Recall that $k$ and $h$ are the uniform mesh spacings in time and space. Besides the usual conservative formulation (1.4), written in terms of the discrete flux $F_k^n$, we suppose that the numerical method can also be written in *viscous form* as

$$u_k^{n+1} = u_k^n - \frac{k}{2h}\left(g_{k+1}^n - g_{k-1}^n\right) + \frac{1}{2}\Big[\Delta\left(Q_{k-1/2}^n \Delta u_{k-1/2}^n\right)\Big], \tag{1.1}$$

where

$$Q_{k+1/2}^n = \frac{k}{h} \cdot \frac{g_k^n + g_{k+1}^n - 2F_{k+1/2}^n}{\Delta u_{k+1/2}}, \tag{1.2}$$

$g_{k+1}^n$ is the *modified flux*, and we have used the convention $\Delta p_{k+1/2}^n = p(u_{k+1}^n) - p(u_k^n)$ for any function $p$. Moreover, assume that the scheme (1.1) satisfies a discrete entropy inequality. Schemes for which such a condition holds are called E-schemes and are at most first order [**37**].

Consider a point $\bar{x}$ at a time $T = Nk$ and a neighborhood of radius $R$ around $\bar{x}$. If $\omega = \sup_u |f'(u)|$, where $f$ is the flux in (1.1), then we determine the grid points at the $n$-th time level inside the domain of dependence of $|x - \bar{x}| < T$ as

$$D^n = \Big\{ j \Big| |x_j - \bar{x}| < R + \omega(T - t_n) \Big\}. \tag{1.3}$$

The bound on the error in the neighborhood of $\bar{x}$ is written as

$$\begin{aligned} \int_{|x-\bar{x}|<R} \big|u(x,T) - v(x,T)\big| \, dx \le & \int_{|x-\bar{x}|<R+\omega T} \big|u(x,0) - v(x,0)\big| \, dx \\ & + C_1 \sup_{n=0,\dots,N} E_n \\ & + C_2 \left[ \sum_{n=0}^{N} \Delta t_{n+1/2} E_n \right]^{1/2}, \end{aligned} \tag{1.4}$$

for constants $C_i$, and with the abbreviation,

$$E_n = \sum_{k \in D^n} \Delta x_{k+1/2} Q_{k+1/2}^n |u_{k+1}^n - u_k^n|.$$

The right hand side of (1.4) is controlled by the quantity $E_n$. Since $0 \leq Q^n_{k+1/2} \leq 1$, each term in the sum $E_n$ is of the form

$$(\Delta x)^2 \frac{\partial u}{\partial x}. \tag{1.5}$$

The global estimate (1.4) therefore depends on local gradient estimators. The difficulty with this error estimate is that a large amount of information has been lost in the constant $C_i$ appearing in (1.4). The estimates of Section 2 avoid this weakness by computing local stability factors and pairing these closely with the local residuals.

The second estimate constructed from Kuznetsov's theory belongs to Kröner and Ohlberger [**30**]. Their estimate applies to multidimensional scalar conservation laws and controls the error in both space and time. We now assume that the approximate solution has been computed with the help of a first-order finite-volume method and we follow the notation introduced in Section 1. Let $R \subset \mathbb{R}^n \times \mathbb{R}^+$ denote the region over which the error is to be estimated and suppose that $R$ is contained inside the cone

$$C = \Big\{ (x,t) \Big| |x - \bar{x}| < \omega(T - t) \Big\}. \tag{1.6}$$

Let $D^n$ denote the set of elements intersecting the cone at time $t_n$,

$$D^n = \Big\{ \text{elements belonging to } C^n \cap \{t = t_n\} \Big\}. \tag{1.7}$$

With this notation, the error bound is

$$\begin{aligned} \int_R \big|u - v\big| \; dxdt \leq & T \Bigg[ C_0 \int_{C \cap \{t=0\}} \big|u(x,0) - v(x,0)\big| \; dx \\ & + C_1 E + C_2 \sqrt{E} \Bigg], \end{aligned} \tag{1.8}$$

where

$$\begin{aligned} E =& \sum_n \sum_{\substack{\text{elements} \\ j \text{ in } D^n}} k_n |\kappa_j| \big|u_j^{n+1} - u_j^n| \\ & + 2 \sum_n \sum_{\substack{\text{edges of elements} \\ j \text{ in } D^n}} k_n \big(k_n + |\kappa_j \bigcup \kappa_l|\big) \Big[ \max_{u_l^n \leq c \leq d \leq u_j^n} \big\{ F^n_{jl}(d,c) - F^n_{jl}(d,d) \big\} \\ & \qquad + \max_{u_l^n \leq c \leq d \leq u_j^n} \big\{ F^n_{jl}(d,c) - F^n_{jl}(c,c) \big\} \Big]. \end{aligned} \tag{1.9}$$

This error bound leads to the following local error estimators

$$\eta_j^0 = \int_{T_j} \big|u(x,0) - v(x,0)\big| \, dx \tag{1.10}$$

$$\eta_j^t = |\kappa_j|^2 \big|u_j^{n+1} - u_j^n\big| \tag{1.11}$$

$$\eta_{jl}^x = (k_n + |\kappa_j \cap \kappa_l|)|\kappa_j \cap \kappa_l|\big|u_j^n - u_l^n|. \tag{1.12}$$

The error can be reduced over the region $R$ by selecting appropriate upper bounds on the the above local error estimators. Examining the estimators (1.11) and (1.12) one can interpret them as gradient indicators weighted by the mesh spacings. For the same reasons mentioned earlier, these estimators perform no better than the indicators of Section 3.

## 2. Error Estimation by Duality Techniques

In this section we discuss a recent technique for the estimation of errors within finite element approximations for hyperbolic problems. This technique has its origin in the work of Babuška and Rheinholt on elliptic two-point boundary value problems and their subsequent extension to other elliptic and parabolic problems [**4**]. These error estimators have been applied mostly to the streamline-diffusion method [**26, 47**] and the discontinuous Galerkin method [**6, 20**].In particular, these error estimators have not been constructed for either lagrangian finite element approximations ALE methods.

Contrary to our error indicators from Chapter 3, these error estimators cannot be used concurrently with the calculation of the approximate solution. In fact, only after both the approximate solution and the solution to an auxiliary problem have been computed, can the error estimator be evaluated. Given this information, the mesh of the approximate solution can then be refined and a second approximate solution can be computed. This process is repeated until the estimated error is below an apriori bound.

**2.1. Theory.** Duality-based error estimators are constructed from a few key ingredients; the weak forms for the exact and approximate solutions, the residual of the approximation, a functional of the solution, and the resulting dual problem. We give an outline of the construction of the error estimator for a nonlinear problem. The presentation is taken from [**23**] and is only meant to indicate the main issues. The evaluation and effectiveness of these estimators will be treated in the next section.

Consider the problem

$$\operatorname{div} \mathcal{F}(u) = 0, \tag{2.1}$$

for $u : \Omega \to \mathbb{R}^n$ with $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$ smooth. Along the boundary of the domain $\Omega$ we assume that

$$B^-(\vec{n})(u - g) = 0, \tag{2.2}$$

where we have used the decomposition into outflow and inflow along the boundary,

$$\begin{aligned} \sum_{i=1}^{n} n_i \mathcal{F}_i(u) &= S^{-1} D S \\ &= S^{-1} D^+ S + S^{-1} D^- S \\ &\equiv B^+(\vec{n}) + B^-(\vec{n}). \end{aligned}$$

Here, $D^+$ and $D^-$ are the positive and negative parts of the diagonal matrix $D$, and $\vec{n} = (n_1, \ldots, n_n)$ is the outwards normal along the boundary. We note that the problem (2.1) covers steady as well as unsteady flows. In weak form, the problem is to find $u$ such that, for all test functions $\phi$ satisfying both $B^+(\vec{n})\phi = 0$ along $\partial\Omega$ and

$$G(u, \phi) \equiv - \int_\Omega \mathcal{F}(u) \cdot \nabla \phi \, dx + \int_{\partial\Omega} B^-(\vec{n}) g \phi \, ds = 0. \tag{2.3}$$

The approximate solution $u_h$ can also be expressed in terms of a pairing but the pairing depends on the numerical method. For definiteness, suppose that $u_h$ is the solution of the usual Galerkin method for a finite element space $S_h$ composed of continuous piecewise linear functions. The approximate solution $u_h$ is the unique solution to

$$\begin{aligned} G_\epsilon(u_h, \phi_h) \equiv &- \int_\Omega \mathcal{F}(u_h) \cdot \nabla \phi_h \, dx + \int_{\partial\Omega} B^-(\vec{n}) g \phi_h \, ds \\ &+ \epsilon \int_\Omega \nabla u_h \cdot \nabla \phi_h \, dx = 0, \qquad \forall \phi \in S_h. \end{aligned} \tag{2.4}$$

We have incorporated the boundary condition $B^+(\vec{n})\phi_h = 0$ into the test space and chosen $\epsilon = Ch|\operatorname{div} \mathcal{F}(u_h)|$. Notice that $G_\epsilon(u, \phi_h) = 0$ since $\epsilon = 0$ when $u$ is a solution to (2.1).

The problem is to measure the error of some functional $J$ of the solution $u$, for example

$$J(u) = \int_\Omega u^T \zeta \, dx \qquad \text{or} \qquad J(u) = \int_{\partial\Omega} u^T A \vec{n} \, dx. \tag{2.5}$$

The functional $J$ could also be nonlinear, like drag over an airfoil. By linearizing both $J$ and $G_\epsilon$, we can define the functionals $L$

$$
\begin{aligned}
J(u) - J(u_h) &= \int_0^1 J'[\theta u + (1-\theta)u_h](u - u_h)\ d\theta \\
&= L(u, u_h; u - u_h)
\end{aligned} \tag{2.6}
$$

and $M$

$$
\begin{aligned}
G_\epsilon(u, \phi) - G_\epsilon(u_h, \phi) &= \int_0^1 G'_\epsilon[\theta u + (1-\theta)u_h](u - u_h)\phi\ d\theta \\
&= M(u, u_h; u - u_h, \phi).
\end{aligned} \tag{2.7}
$$

The prime in (2.6) and (2.7) denotes the Fréchet derivative. Replacing $u - u_h$ in these expressions by $w$, we construct the following dual problem; find $z$ such that

$$
M(u, u_h; w, z) = L(u, u_h; w) \qquad \forall w. \tag{2.8}
$$

We remark that the existence of a solution $z$ is still an open problem, but we will return to this issue later. Using the fact that $G_\epsilon(u, \phi_h) - G_\epsilon(u_h, \phi_h) = 0$, we obtain the error representation formula

$$
\begin{aligned}
J(u) - J(u_h) =& L(u, u_h; u - u_h) \\
=& M(u, u_h; u - u_h, z) \\
=& M(u, u_h; u - u_h, z - z_h) \\
=& - G_\epsilon(u_h, z - z_h).
\end{aligned} \tag{2.9}
$$

We now apply this procedure to our problem (2.3) and (2.4) with the first functional in (2.5).

$$
\begin{aligned}
J(u) - J(u_h) =& \int_\Omega \mathcal{F}(u_h) \cdot \nabla(z - z_h)\ dx - \int_{\partial\Omega} B^-(\vec{n})g(z - z_h)\ ds \\
& - \epsilon \int_\Omega \nabla u_h \cdot \nabla(z - z_h)\ dx \\
=& - \int_\Omega \operatorname{div} \mathcal{F}(u_h)(z - z_h)\ dx \\
& + \int_{\partial\Omega} \Big[\vec{n} \cdot \mathcal{F}(u_h) - B^-(\vec{n})g\Big](z - z_h)\ ds \\
& - \epsilon \int_\Omega \nabla u_h \cdot \nabla(z - z_h)\ dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\Omega} r_h(z - z_h)\ dx + \int_{\partial\Omega} \sigma_h(z - z_h)\ ds \\
&\quad - \epsilon \int_{\Omega} \nabla u_h \cdot \nabla(z - z_h)\ dx
\end{aligned} \tag{2.10}
$$

We have introduced the residuals $r_h$ and $\sigma_h$ which measure the errors in respectively the interior and boundary of the domain. Rewriting the integrals over the domain $\Omega = \bigcup_{\kappa \in \mathcal{T}} \kappa$ into a sum of integrals over the cells $\kappa$, we find an error estimate in terms of local error estimators

$$
\begin{aligned}
J(u) - J(u_h) &= \int_{\Omega} r_h(z - z_h)\ dx + \int_{\partial\Omega} \sigma_h(z - z_h)\ ds \\
&\quad - \epsilon \int_{\Omega} \nabla u_h \cdot \nabla(z - z_h)\ dx \\
&\equiv \sum_{\kappa \in \mathcal{T}} \eta_\kappa
\end{aligned}
$$

Examining (2.10), one can see that the dual problem provides weights for the various residual terms. In this sense, the dual problem describes the influence of the residuals (the source of the error) on the quality of the output functional $J$. Similar a posteriori error estimates may be obtained for other hyperbolic problems, finite element methods, and functionals. Semi-discrete approximations can also be analyzed but the treatment of boundary conditions requires particular attention.

**2.2. Practice.** The formal procedure outlined in Section 2.1 is straightforward to implement, unfortunately difficulties arise with the definition and solution of the dual problem (2.8). For linear equations, the dual problem is well-posed while for nonlinear problems (without viscosity), the dual problem is linear with discontinuous coefficients. In this second case, the existence and stability theory of such problems is unknown. Researchers nonetheless continue to construct these error estimators by introducing approximations which permit the computation of a dual problem.

Johnson and Süli advocate the use of a stabilized finite element method which contains a small amount of viscosity (like the $\epsilon$ term in (2.4) ). This term also stabilizes the dual problem (2.8) and allows the computation of an approximation $z^*$ to the exact solution $z$. Since only the difference $z - z_h$ are eventually needed in (2.10), the difference is replaced by

$$
z - z_h \approx z^* - \pi_h z^*, \tag{2.11}
$$

where $\pi_h$ is the projection onto the finite element space $S_h$. This approximation is only valid if $z^*$ contains a significant component in $S_h^\perp$.

To accomplish this, $z^*$ can be computed with higher order elements and/or a new mesh (usually one order higher or a new mesh with 50% more elements). The dual problem (2.8) as stated depends on the exact solution and therefore must also be approximated. The most natural choice is to replace $u$ by $u_h$ in (2.8) although the effect of this change depends on the stability of the dual problem. This source of error is difficult to assess for nonlinear problems.

Experiments have shown that these error estimators are reliable and efficient. The local error estimators $|\eta_\kappa|$ are generally within 1-10% of the true error [**23**] and lead to mesh refinement along characteristics, not along shock fronts like the ad-hoc error indicators of Section 3.

One drawback with this approach is the high cost of the iterations required for an acceptable approximation. This occurs because the error in an approximation $u_h$ cannot be evaluated until both the primal (2.1) and the dual problem (2.8) have been solved. For a given `tolerance`, an approximate solution $u_h$ satisfying

$$|J(u) - J(u_h)| < \texttt{tolerance}, \tag{2.12}$$

can be found by following the procedure below.

$\texttt{compute } u_h\texttt{, } z^*\texttt{, } \eta_\kappa$
$\texttt{while } \sum \eta_\kappa > \texttt{tolerance}$
$\quad\texttt{refine cells } \kappa \texttt{ where } \eta_\kappa \texttt{ large}$
$\quad\texttt{compute } u_h\texttt{, } z^*\texttt{, } \eta_\kappa$

Although $z^*$ is solved on a finer mesh than $u_h$, the dual problem is inexpensive since it is a linear problem. This technique provides the most efficient approximate solution for a fixed tolerance, but the required intermediate solutions raise the overall cost of the process.

CHAPTER 5

# Conclusion

Error estimation for hyperbolic problems is an active field of research that has begun to attract attention since the work of Berger and Collela [**8**], Johnson and Szepessy [**26**], and Houston et al. [**21**]. The techniques in error estimation can be broadly classified as either heuristic or rigorous, depending on whether or not they have any demonstrated relationship to the error. For heuristic error indicators, we have found few comparative studies of their performance and therefore, have proposed general guidelines for their assessment. On the other hand, for rigorous error estimators we have not identified any methods that could be immediately applicable to either the structured mesh component or to the ALE component of ALEGRA. Despite these observations, we make some recommendations on techniques that could be adapted to Sandia National Laboratory's ALEGRA code.

In our survey, we compared error indicators according to the following conditions

A) it should vanish under mesh refinement;
B) it should measure the errors in both the smooth and non-smooth regions;
C) it should measure the transmitted error;
D) it should be inexpensive.

The first three conditions require that the error indicator behave like the true error.

The class of gradient-based indicators satisfy only the last of these conditions and unless cost is an important issue, are not recommended. A second class of indicators is based on the residual of the approximation. The residual has some theoretical support since it appears weighted in error estimators based on duality techniques. When computed in an appropriate norm and when smoothed, it can measure the local error [**23**] and detect errors within smooth regions [**54**], for a cost comparable to gradient-based indicators. Methods based on comparing the original approximation to a reconstructed approximation fail when the reconstruction depends only on local information. Examples are known where indicators based on reconstructions of the approximation fail to satisfy conditions B) and C). Of all the heuristic error indicators, Richardson extrapolation is the only method satisfying all four criteria. The computational overhead of this method is manageable only when applied within Berger and Collela's AMR framework [**8**]. Numerous studies [**54, 56**] have shown its superiority over other error indicators and in practice its error estimate are often close to the true error.

A general framework for the development of reliable error estimators for hyperbolic problems has been introduced by several researchers [**20, 22, 26**]. These estimators satisfy all four conditions although their performance is slightly hindered by the presence of discontinuities. Notwithstanding the fact that these techniques have not yet been applied to lagrangian finite element methods, it is believed that this framework can be adapted to ALEGRA. Applications of these methods to nonlinear and time-dependent problems are being studied [**23**]. Generally speaking, duality techniques for error estimation offer the best avenue of development for ALEGRA. As a caveat, we mention that these techniques cannot be used concurrently with the computation of the approximation. They require adaptively calculating a sequence of approximations.

The situation can therefore be summarized in terms of the computational effort one is willing to bear. For problems where the true error is not of interest, refinement can be controlled by the residual measured in the $L^1$-norm or the residual of a smoothed approximation [**28, 48**]. The method of error estimation that offers the best compromise between cost and accuracy is Richardson extrapolation, at least when used with nested grids [**8**]. Finally the most promising error estimators are based on duality techniques and can only be constructed for the certain finite element methods. As these error estimators are further studied, it is hoped that they can come to be routinely applied to a wider variety of nonlinear problems and numerical methods.

# Bibliography


[1] A. Adjerid, J. E. Flaherty, and I. Babuška. A posteriori error estimation for the finite element method-of-lines solution of parabolic problems. *Comput. Methods Appl. Mech. Engrg.*, 9:261–286, 1999.

[2] M. J. Aftomis. Upwind method for simulation of viscous flow on adaptively refined meshes. *AIAA Journal*, 32(2):268–277, 1994.

[3] M. J. Aftomis, M. J. Berger, and J. E. Melton. Adaptive cartesian mesh generation. In J. F. Thompson, B. Soni, and N. P. Weatherrill, editors, *Handbook of Grid Generation*. CRC Press, 1998.

[4] M. Ainsworth and J. T. Oden. *A Posteriori Error Estimation in Finite Element Analysis*. John Wiley & Sons, New York, 2000.

[5] T. Barth. Aspects of unstructured grids and finite-volume solvers for the euler and navier-stokes equations. In *von Karman Lecture Series*. 1994. 1994-05.

[6] T. Barth and M. G. Larson. A posteriori error estimation for discontinuous galerkin approximations of hyperbolic systems. In *Lecture Notes in Computational Sciences and Engineering*, volume 11. Springer-Verlag, 2000.

[7] J. Bell, M. Berger, J. Saltzman, and M. Welcome. Three-dimensional adaptive mesh refinement for hyperbolic conservation laws. *SIAM J. Sci. Comput.*, 15(1):127–138, 1994.

[8] M. Berger and P. Colella. Local adaptive mesh refinement for shock hydrodynamics. *J. Comp. Phy.*, 82:64–84, 1989.

[9] M. J. Berger and R. J. LeVeque. Adaptive mesh refinement using wave-propagation algorithms for hyperbolic systems. *SIAM J. Numer. Anal.*, 35(6):2298–2316, 1998.

[10] M. Berzins. Mesh quality: A function of geometry, error estimates or both? *Engr. Comp.*, 15:236–247, 1999.

[11] M. A. Christon and A. C. Robinson. High-resolution algorithms for conservation laws as a generalized simulation framework. Proposal to mics program, Sandia National Laboratories, 2001.

[12] M. J. Coirier. An adaptively-refined, cartesian, cell-based scheme for the euler equations. *NASA TM-106754*, 1994.

[13] C. Dafermos. *Hyperbolic Conservation Laws in Continuum Pysics*, volume 325 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York, 2000.

[14] J. F. Dannenhofer and J. R. Baron. Grid adaptation for the 2-d euler equations. Aiaa 85-0484, 1985.

[15] B. DeVolder, J. Glimm, J. W. Grove, Y. Kang, Y. Lee, K. Pao, D. H. Sharp, and K. Ye. Uncertainty quantification for multiscale simulations. *Journal of Fluids Engineering*, 2001. Submitted; Los Alamos National Laboratory

Preprint LA-UR 01-4022 SUNY Stony Brook Preprint SUNYSB-AMS-2001-08.

[16] B. Engquist and B. Sjogreen. The convergence rate of finite difference schemes in the presence of shocks. *SIAM J. Numer. Anal*, 35:2464–2485, 1998.

[17] K. O. Friedrichs. Symmetric positive linear differential equations. *Comm. Pure Appl. Math.*, 11:333–418, 1958.

[18] O. Friedrichs. Weighted essentially nonoscillatory schemes for the interpolation of mean values on unstructured grids. *J. Comput. Phys.*, 144:194, 1998.

[19] L. Gosse and C. Makridakis. Two a posteriori error estimates for one-dimensional scalar conservation laws. *SIAM J. Numer. Anal.*, 38(3):964–988, 2000.

[20] R. Hartmann and P. Houston. Adaptive discontinuous galerkin finite element methods for nonlinear hyperbolic conservation laws. *SIAM J. Sci Comp.*, 17(1-4):979–1004, 2002.

[21] P. Houston, J. A. Mackenzie, E. Süli, and G. Warnecke. A posteriori error analysis for numerical approximations of Friedrichs systems. *Numer. Math.*, 82:433–470, 1999.

[22] P. Houston, C. Schwab, and E. Süli. hp-finite element method for hyperbolic problems. Technical report no. 11, Oxford University, 1999.

[23] P. Houston and E. Süli. Adaptive finite element approximation of hyperbolic problems. Technical report no. 1, Oxford University, 2002.

[24] C. Ilinca, X.D. Zhang, J.-Y. Trépanier, and R. Camarero. A comparison of three error estimation techniques for finite-volume solutions of compressible flows. *Comput. Methods Appl. Engrg.*, 189:1277–1294, 2000.

[25] C. Johnson. Adaptive finite elements for conservation laws. In A. Quarteroni, editor, *Advanced Numerical Approximation of Nonlinear Hyperbolic Equations*, volume 1697 of *Lecture Notes in Mathematics*, pages 269–323. Springer-Verlag, Berlin, 1998.

[26] C. Johnson and A. Szepessy. Adaptive finite element methods for conservation laws based on a posteriori error estimates. *Comm. Pure Appl. Math.*, 48:199–234, 1995.

[27] Y. G. Kallinderis and J. R. Baron. Adaptation methods for a new navier-stokes algorithm. *AIAA J.*, 27:449–466, 1985.

[28] S. Karni, A. Kurganov, and G. Petrova. A smoothness indicator for adaptive algorithms for hyperbolic systems. *J. Comp. Phy.*, 178:323–341, 2002.

[29] D. Kröner. *Numerical Schemes for Conservation Laws.* Wiley & Teubner, Chichester – Stuttgart, 1997.

[30] D. Kröner and M. Ohlberger. A posteriori error estimates for upwind finite volume schemes for nonlinear conservation laws in multi dimensions. *Math. Comput.*, 69(229):25–39, 1999.

[31] N. N. Kuznetsov. Accuracy of some approximate methods for computing the weak solutions of a first-order quasi-linear equation. *USSR Comp. Math. and Math. Phys.*, 16(6):105–119, 1967.

[32] M. Laforest. *A posteriori error estimate for front-tracking.* PhD thesis, State University of New York at Stony Brook, 2001.

[33] M. Laforest. A second-order residual indicator over structured meshes. Technical report, Colorado State University, 2002.

[34] T. P. Liu and T. Yang. $L_1$ stability for $2 \times 2$ systems of hyperbolic conservation laws. *J. of AMS*, 12(3):729–774, 1999.

[35] H. Luo, J. D. Baum, and R. Löhner. An efficient spatial adaption algorithm on 3-d unstructured meshes for the euler equations. 1993.

[36] H. Nessyahu and E. Tadmor. The convergence rate of approximate solutions for nonlinear scalar conservation laws. *SIAM J. Numer. Anal.*, 29:1505–1519, 1992.

[37] S. Osher and E. Tadmor. On the convergence of difference approximation to scalar conservation laws. *Math. Comput.*, 50:19–51, 1988.

[38] J. Peraire, M. Vahdati, K. Morgan, and O. C. Zienkiewicz. Adaptive remeshing for compressible flow computations. *J. Comput. Phy.*, 72:449–466, 1987.

[39] S. Prudhomme. Adaptive finite element simulation of a supersonic underexpanded jet. Masters thesis, University of Virginia, 1992.

[40] G. Puppo. Numerical entropy production on shocks and smooth transitions. *J. Sci. Comp.*, 17(1-3):287–296, 2002.

[41] R. D. Rausch, J. T. Batina, and H. T. Y. Yang. Spatial adaptation procedures on unstructured meshes for accurate unsteady aerodynamic flow computation. Aiaa 91-1106, 1991.

[42] P. J. Roache. *Verification and Validation in Computational Science and Engineering*. Hermosa Publishers, New Mexico, 1998.

[43] C. J. Roy. Grid convergence error analysis for mixed-order numerical schemes. Aiaa paper 2001-2606, 2001.

[44] T. Sonar. Strong and weak norm refinement indicators based on the finite element residual for compressible flow computations. *Impact of Comput. Sci. Engr.*, 5:111–127, 1993.

[45] T. Sonar and E. Süli. A dual graph-norm refinement indicator for finite volume approximations of euler equations. *Numer. Math.*, 78:619–658, 1998.

[46] R. Struijs, P. Vankeirsbilck, and H. Deconinck. An adaptive grid polygonal finite volume method for the compressible flow equations. Technical Report 89-1959-CP, 1989.

[47] E. Süli. A-posteriori error analysis and adaptivity for finite element approximations of hyperbolic problems. In M. Ohlberger D. Kröner and C. Rohde, editors, *An Introduction to Recent Developments in Theory and Numerics for Conservation Laws*, volume 5 of *Lect. Notes Comput. Sci. Eng.*, pages 123–194. Springer, 1999.

[48] I. Thomas and T. Sonar. On a second-order residual estimator for numerical schemes for nonlinear hyperbolic conservation laws. *J. Comp. Phy.*, 171:227–242, 2001.

[49] J. F. Thompson, B. Soni, and N. P. Wheatherrill. *Handbook of Grid Generation*. CRC Press, 1998.

[50] J. Trangenstein. Adaptive mesh refinement for wave-propagation in nonlinear solids. *SIMA Sci. Comput.*, 16(4):819–839, 1995.

[51] John Trangenstein. A second-order godunov algorithm for two-dimensional solid mechanics. *Comput. Mech.*, 13:343–359, 1994.

[52] G. P. Warren, W. K. Anderson, J. L. Thomas, and Sherrie L. Krist. Grid convergence for adaptive methods. 1991. AIAA 91-1592.

[53] G. Zakeri. Resolution function and adaptive refinement method for a system of conservation laws. In *Fifth Copper Mountain Conference on Multigrid Methods*, 1991.

[54] X. D. Zhang, D. Pelletier, J.-Y. Trépanier, and R. Camarero. Numerical assessment of error estimators for euler equations. *AIAA Journal*, 39(9):1706–1715, 2001.

[55] X. D. Zhang, J.-Y. Trépanier, and R. Camarero. An a posteriori error estimation method based on error equations. 1997. AIAA 1997-1889.

[56] X. D. Zhang, M.-G. Vallet, J. Dompierre, P. Labbé, D. Pelletier, J.-Y. Trépanier, R. Camarero, J. V. Lassaine, L. M. Manzano, , and D. W. Zingg. Mesh adaptation using different error indicators for the euler equations. In *15th AIAA CFD Conference*, number AIAA 2001-2549, 2001.

[57] O. C. Zienkiewicz and J. C. Zhu. A simple error estimator and adaptive procedure for practical engineering analysis. *Int. J. Numer. Meth. Engrg.*, 24:337–357, 1987.