A Posteriori Error Estimate for Front-Tracking

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Abstract of the Dissertation

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We demonstrate an a posteriori error estimate in the L^1 norm for front-tracking approximate solutions to hyperbolic systems of nonlinear conservation laws. Extending the L^1-stability result of Bressan, Liu, and Yang we use their L^1-equivalent functional for pairs of front-tracking approximations and identify the leading order contribution to the numerical error. This leading term is closely related to the residual of the approximation and determines an a posteriori bound of the error. We demonstrate the estimate for the front-tracking approximations of Risebro, which are extensions to systems of Dafermos' polygonal approximations.

Patrick Royer

1971-2000

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Chapter 1

Conservation Laws and Error Estimates

We begin the first chapter with a review of the theory of hyperbolic conservation laws. In Section 1.2, we introduce a posteriori error estimation and survey the literature of such estimates for conservation laws.

1.1 Theory of Hyperbolic Conservation Laws

The basic physical principles of conservation of mass, momentum, and energy find their mathematical expression in systems of conservation laws. These principles state that for an extensive quantity throughout a domain, the total amount produced within the domain must balance the flux through the boundary of the domain. For a domain $\mathcal{D}$, where P is the amount produced, $\vec{F}$ is the flux, and $\vec{n}$ is the outward unit normal to the boundary, a conservation law can be expressed as

$$\int_{\mathcal{D}} P(x)\ dx = \int_{\partial\mathcal{D}} \vec{F}(x) \cdot \vec{n}\ dS. \tag{1.1.1}$$

Depending upon the physics to be modeled, this framework can give rise to equations of almost any type; elliptic, parabolic, hyperbolic, or a mix thereof.

In this work, we focus upon equations which arise in dynamical systems without dissipative mechanisms. Among the physical systems which we have in mind we note magneto-hydrodynamics, balance laws in general relativity, chromatography, and most importantly, isentropic gas dynamics as modeled by Euler's equations. For these systems, there exists a flux function f, whose Jacobian has real and distinct eigenvalues, such that at every point of space $x \in \mathbb{R}$ and time $t \in \mathbb{R}^+$, the state of the system $u \in \mathbb{R}^n$ satisfies

$$u_t + f(u)_x = 0. \tag{1.1.2}$$

Under our assumptions, equation (1.1.2) is the differential form of equation (1.1.1). Given the initial state of the system $u(0, x)$, the basic problem is to describe the

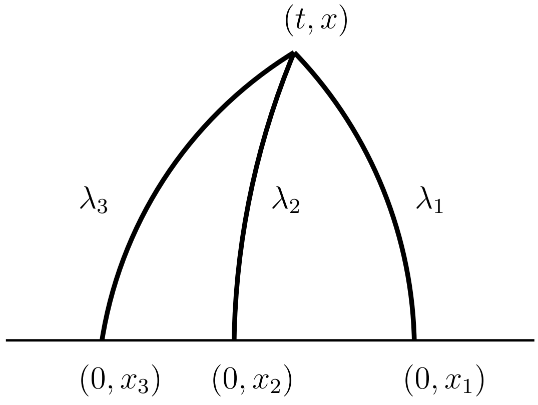

Figure 1.1: Intuitively, the solution at (t,x) is the sum of the k-th components of the initial data at $(0,x_k)$ propagated along curves with velocities λ_k.

qualitative and quantitative features of the system u as it evolves over time following equation (1.1.2).

Henceforth, we shall consider only systems of equations satisfying the strict hyperbolicity condition: the real eigenvalues $\lambda_1(u) < \ldots < \lambda_n(u)$ of the Jacobian $Df(u)$ must be distinct in the neighborhood of the origin in $\Omega \subset \mathbb{R}^n$ where the smooth flux is defined. Intuitively speaking, it would seem reasonable that strict hyperbolicity imply, at least for smooth initial data and a short time t, that the problem (1.1.2) is well-posed. Since the eigenvalues are distinct one could attempt to diagonalize the system. One might expect that, up to a small perturbation, the solution at a later time t could be reconstructed by propagating the k-th component, in the basis of eigenvectors, of the initial data at $(0,x_k)$ along a curve traveling with speed λ_k up to (t,x). This technique is called the method of characteristics and is illustrated in Figure 1.1.

In fact, solutions to equation (1.1.2) spontaneously develop singularities within a finite amount of time, no matter how smooth or how small the initial data [25]. The singularities developed can be discontinuities in the solution, its derivative, or even locally unbounded growth in the L^∞-norm [24, 27]. Under these circumstances, equation (1.1.2) fails in the classical sense and one must look for solutions in the distributional sense. Given initial data $\bar{u} : \mathbb{R} \longrightarrow \Omega$ we say that the bounded measurable function $u : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \Omega$ is a weak solution of the system (1.1.2) if $u(0,x) = \bar{u}(x)$ and if

$$\int_0^\infty \int_{-\infty}^\infty \big(\phi_t u + \phi_x f(u)\big)\ dxdt + \int_{-\infty}^\infty \phi(x,0)u(x,0)\ dx = 0, \tag{1.1.3}$$

for every smooth ϕ with compact support in $0 \leq t$.

Discontinuities in weak solutions to (1.1.2) are an expression of the nonlinearity

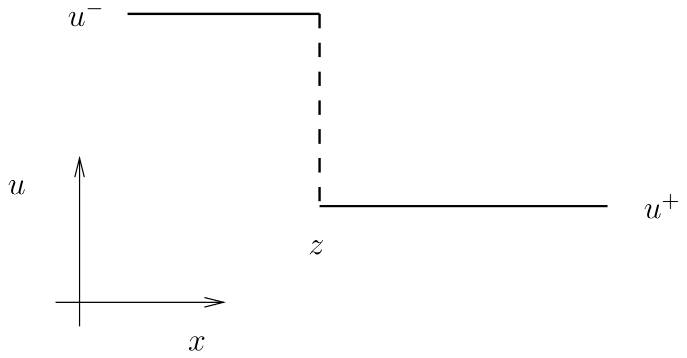

Figure 1.2: Illustration of a general shock discontinuity satisfying the Rankine-Hugoniot condition.

of f and as such, are an important element in the study of these equations. In gas dynamics, they correspond to shock waves near the front of supersonic aircraft, on highways they correspond to traffic jams, and in oil reservoirs to sudden decreases in effective oil recovery.

Consider now a discontinuity located along the smooth curve $\big(t, z(t)\big)$ and separating two smooth regions of the solution u. Suppressing the time dependence, let

$$u^{\pm} = \lim_{x \to z(t)\pm} u(t, x). \tag{1.1.4}$$

Using the integral form of the conservation law (1.1.1) in a small neighborhood of $\big(t, z(t)\big)$ containing the support of a smooth test function ϕ, we obtain

$$f(u^+) - f(u^-) = \dot{z}\big(u^+ - u^-\big). \tag{1.1.5}$$

This relationship between the speed of propagation of the discontinuity and the neighboring states of the discontinuity is called the *Rankine-Hugoniot* condition. Figure 1.2 represents such a discontinuity.

We shall assume that each family $k \in \{1, \dots, n\}$ is either *genuinely nonlinear*

$$r_k \cdot \nabla \lambda_k \neq 0, \tag{1.1.6}$$

or *linearly degenerate*

$$r_k \cdot \nabla \lambda_k = 0. \tag{1.1.7}$$

For a scalar conservation law, condition (1.1.6) reduces to $f'' \neq 0$ and (1.1.7) reduces to $f'' = 0$. Let $r_1, \ldots, r_n$ denote the smooth eigenvectors corresponding to $\lambda_1, \ldots, \lambda_n$. We fix the normalization of the eigenvector r_k belonging to a genuinely nonlinear family by requiring that

$$r_k \cdot \nabla \lambda_k = 1. \tag{1.1.8}$$

With this normalization, we now parameterize the pairs of states u^- and u^+ satisfying (1.1.5). Given the matrix

$$A(u^-, u^+) = \int_0^1 Df\big((1-\theta)u^- + \theta u^+\big)\, d\theta, \tag{1.1.9}$$

it is easy to verify that condition (1.1.5) is equivalent to

$$A(u^-, u^+)(u^+ - u^-) = \dot{z}(u^+ - u^-). \tag{1.1.10}$$

This implies that there exists an integer k such that $\dot{z}$ is the k-th eigenvalue of $A(u^-, u^+)$ and $(u^+ - u^-)$ is an associated eigenvector. Let $s_k(u^-, u^+)$ denote the k-th eigenvalue of $A(u^-, u^+)$, also called the Rankine-Hugoniot shock speed. Note that when (1.1.2) is a scalar equation, $n = 1$, the Rankine-Hugoniot shock speed can be computed directly from the scalar relation (1.1.5). Omitting the subscript $k = 1$, for a scalar conservation law we have

$$s(u^-, u^+) = \frac{f(u^-) - f(u^+)}{u^- - u^+}. \tag{1.1.11}$$

Definition 1.1.1 *The k-th shock curve $u^+ = S_k(\sigma)(u^-)$ connects the set of states u^- and u^+ satisfying* (1.1.10) *with $\dot{z} = s_k(u^-, u^+)$. For genuinely nonlinear families k, the parameter σ satisfies*

$$\frac{d}{d\sigma}\lambda_k\big(S_k(\sigma)(u^-)\big) = 1, \tag{1.1.12}$$

$$S_k(0)(u^-) = u^-,$$

while for linearly degenerate families, σ is directed arc-length.

Discontinuities separated by states satisfying the Rankine-Hugoniot condition form a family of discontinuous solutions to (1.1.2) but there also exists continuous solutions associated to genuinely nonlinear families. Since equation (1.1.2) is invariant under a scaling of both space and time, we propose the ansatz $u(x,t) = u(\xi)$ with

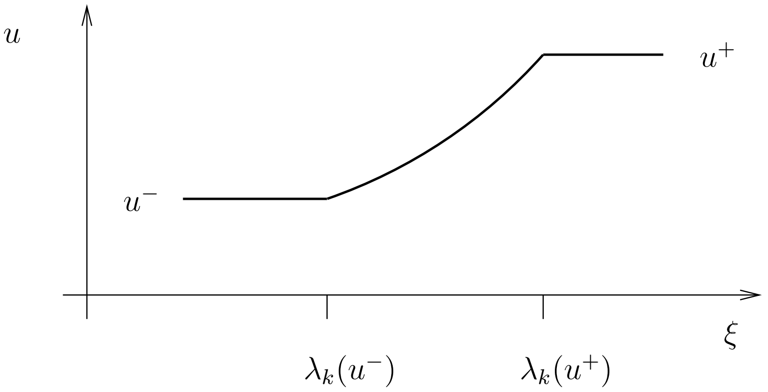

Figure 1.3: Illustration of a continuous rarefaction fan satisfying equation (1.1.13).

$\xi = x/t$. Substituting this identity into (1.1.2), we obtain

$$(I\xi - Df)u_\xi = 0. \tag{1.1.13}$$

This relation states that ξ must be an eigenvalue of Df with eigenvector u_ξ. In other words, along rays traveling at speed ξ the solution u takes on values satisfying

$$\xi = \lambda_k(u).$$

We use these facts to parameterize these continuous solutions.

Definition 1.1.2 *For a given left-hand state u^-, and a genuinely nonlinear family k, we define the rarefaction curve $R_k(\cdot)(u^-)$ to be the solution to*

$$\frac{d}{d\sigma} R_i(\sigma)(u^-) = r_i\big(R_i(\sigma)(u^-)\big), \tag{1.1.14}$$

$$R_i(0)(u^-) = u^-.$$

For a genuinely nonlinear family k, identities (1.1.8),(1.1.12), and (1.1.14) imply that

$$\lambda_k\big(S_k(\sigma)(u)\big) - \lambda_k(u) = \sigma, \tag{1.1.15}$$

$$\lambda_k\big(R_k(\sigma)(u)\big) - \lambda_k(u) = \sigma.$$

As is well-known, from even these simple solutions one can exhibit non-unique solutions to initial-value problems of (1.1.2). Mathematically, this issue is related to a lack of stability although fundamentally the difficulty originates from the omission of certain physics from the original model. In general, this problem is handled by introducing a physical quantity, the entropy, and appealing to the second law of thermodynamics which states that entropy must increase across all waves. We formalize this notion in the following definition.

Definition 1.1.3 *An entropy-entropy flux pair* (η, q) *for the system* (1.1.2) *is a pair of smooth functions over* Ω*, with* η *convex, satisfying*

$$\nabla q = \nabla \eta D f. \tag{1.1.16}$$

A solution u *of system* (1.1.2) *is said to be entropy-admissible if, for all entropy-entropy flux pairs,*

$$\eta(u)_t + q(u)_x \leq 0, \tag{1.1.17}$$

in the sense of distributions.

Although there exists many distinct but related notions of 'entropy' solutions, Definition 1.1.3 will be sufficient for our purposes. Most of these notions coincide for strictly hyperbolic systems with genuinely nonlinear characteristic families. For more information, we refer the reader to Dafermos' treatise [13].

We note that smooth solutions of system (1.1.2) always satisfy the entropy condition (1.1.17). This follows immediately from the compatibility condition (1.1.16) for the entropy-entropy flux pair and the conservation law (1.1.2). One can also verify that shock discontinuities from linearly degenerate families satisfy condition (1.1.17) for all values of the parameter σ. Since characteristics in smooth rarefaction solutions must diverge, relation (1.1.15) implies that the solution $u^+ = R_k(\sigma)(u^-)$ exists only when $\sigma > 0$. For shocks from genuinely nonlinear families, it is well-known that the entropy condition is locally satisfied if and only if $u^+ = S_k(\sigma)(u^-)$ with $\sigma < 0$. These facts suggest combining the shock and rarefaction curves to construct entropy-admissible solutions. In fact, the following lemma shows that these curves are closely related geometrically.

Lemma 1.1.1 *[31] Consider a state* u^-*, an integer* $k \in \{1, \ldots, n\}$*, and a parameter* $\sigma \in \mathbb{R}$*, then for* $u^+ = S_k(\sigma)(u^-)$

$$\big|\big|u^+ - R_k(\sigma)(u^-)\big|\big| = \mathcal{O}(1)|\sigma|^3, \tag{1.1.18}$$

$$\Big|s_k(u^-, u^+) - \frac{1}{2}\big(\lambda_k(u^-) + \lambda_k(u^+)\big)\Big| = \mathcal{O}(1)|\sigma|^2, \tag{1.1.19}$$

where $\mathcal{O}(1)$ *is uniformly bounded for* u^- *and* u^+ *in* Ω.

For a genuinely nonlinear family k define

$$T_k(\sigma)(u^-) = \begin{cases} R_k(\sigma)(u^-), \sigma \geq 0 \\ S_k(\sigma)(u^-), \sigma < 0 \end{cases} \tag{1.1.20}$$

and for a linearly degenerate family let $T_k(\sigma)(u^-) = S_k(\sigma)(u^-)$. According to Lemma 1.1.1, the curve T_k is C^2.

One can now construct a solution to the simplest initial-value problems.

Definition 1.1.4 *A Riemann problem is an initial-value problem for* (1.1.2) *with piecewise constant initial data*

$$u(0,x) = \begin{cases} u^- & , \text{if } x < 0, \\ u^+ & , \text{if } x \geq 0. \end{cases} \tag{1.1.21}$$

Theorem 1.1.1 *[31] For a given value of* $u_0 \in \Omega$, *there exists a neighborhood* $\Omega' \subset \Omega$ *of* u_0 *such that for any* $u^-, u^+ \in \Omega'$, *the Riemann problem has a unique self-similar solution composed of* $n+1$ *constant states* $u_0 = u^-, u_1, \ldots, u_n = u^+$ *satisfying* $u_k = T_k(p_k)(u_{k-1})$ *for real numbers* $p_k = p_k(u^-, u^+)$. *Two consecutive states* u_{k-1} *and* u_k *separate either*

i) a continuous region where the solution $u(x/t)$ *satisfies* $\lambda_k\big(u(x/t)\big) = x/t$ *and*

$$u_k = R_k(p_k)(u_{k-1}),$$

ii) or, a discontinuity traveling with the Rankine-Hugoniot speed and connecting

$$u_k = S_k(p_k)(u_{k-1}).$$

This solution is entropy-admissible.

A similar result also holds with the wave curves T_k replaced by shock curves.

Theorem 1.1.2 *For a given value of* $u_0 \in \Omega$, *there exists a neighborhood* $\Omega' \subset \Omega$ *of* u_0 *such that for any* $u^-, u^+ \in \Omega'$, *there exists a unique sequence of numbers* $q_1(u^-, u^+), \ldots, q_n(u^-, u^+)$ *satisfying*

$$u^+ = S_n\big(q_n(u^-, u^+)\big) \circ \cdots \circ S_1\big(q_1(u^-, u^+)\big)(u^-). \tag{1.1.22}$$

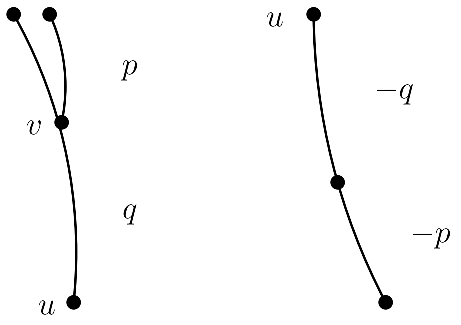

Figure 1.4: Figure illustrating the states in inequalities (1.1.26) and (1.1.27).

The vector $\big(p_1(u^-,u^+),\dots,p_n(u^-,u^+)\big)$ will be called the wave coordinates between u^- and u^+ and $\big(q_1(u^-,u^+),\dots,q_n(u^-,u^+)\big)$ will be called the shock coordinates.

Let Λ be the positive constant satisfying

$$\sup_{\substack{u\in\Omega\\ i=1,\dots,n}} \big|\lambda_i(u)\big| = \Lambda. \tag{1.1.23}$$

We introduce the constant

$$d = \min\{d_1, d_2, d_3\} \tag{1.1.24}$$

depending on the three distinct constants described below. At the cost of possibly reducing the size of Ω, we assume that there exists a positive constant d_1 such that

$$d_1 < \inf\Big\{|l_i - l_j| \,\Big|\, i \neq j \text{ and for } k = i,j,\ l_k \in \lambda_k(\Omega),\ \text{or} \quad l_k = s_k(u,v)\ \forall u,v\in\Omega\Big\}. \tag{1.1.25}$$

The genuine nonlinearity of f implies that for any $u \in \Omega$, any k, and any positive quantities p, q such that both $v \equiv S_k(q)(u)$ and $S_k(p+q)(u)$ belong to Ω, then there exists a positive constant d_2 such that

$$d_2|q| < s_k\big(v, S_k(p)(v)\big) - s_k\big(u, S_k(p+q)(u)\big). \tag{1.1.26}$$

For any $u \in \Omega$, any k, and any positive quantities p, q such that both $S_k(-q)(u)$ and

$S_k(-p-q)(u)$ belong to Ω, then there exists a positive constant d_3 such that

$$d_3|p| < s_k\big(u, S_k(-q)(u)\big) - s_k\big(u, S_k(-p-q)(u)\big). \tag{1.1.27}$$

Figure 1.4 illustrates the states in (1.1.26) and (1.1.27). These constants have been grouped together under d since they will serve the same purpose later in the proof.

This concludes our summary of the theory of hyperbolic conservation laws. Although the theory has become quite sophisticated, in particular in its attempt to characterize solutions from various entropy conditions, the prerequisites of this monograph are limited to the topics of this section. For more information, we refer the interested reader to the treatise of Dafermos [13].

1.2 A Posteriori Error Estimates

In this section, we present an overview of a posteriori error estimation for conservation laws. In this field, the goal is not only to estimate the error in an approximation but to estimate the error without assuming prior knowledge of the exact solution. A posteriori error estimates should be computable entirely from the approximation, the initial data, and the equation. In this sense, they are rigorous and computable estimates which which could be used as global error bounds, criteria for local mesh refinement, and even to quantify modeling errors.

A posteriori error estimates are categorized by the stability theory upon which they are constructed. For the finite element method (FEM), work began with the estimates of Babuška and Rheinboldt [3] for elliptic boundary value problems. Since then, a large number of a posteriori error estimates have been proposed for elliptic and parabolic problems. These estimates depend on the stability of the problem or of its dual, on interpolation estimates, and on Galerkin orthogonality. We defer to the text of Oden and Ainsworth [36] for a recent survey of those techniques. For elliptic and parabolic equations, the theory has now moved towards understanding the reliability and efficiency of such estimates [4].

For hyperbolic equations and in particular for conservation laws, the theory is still in development. Most estimates have been obtained either from Kuznetsov's stability theory [30] or from the strong stability of the dual problem for conservation laws with viscosity [22, 26]. Other a posteriori error estimates for hyperbolic equations have been constructed from the stability theory of Friedrichs [16] and the theory of Tadmor [43]. The a posteriori error estimates of this dissertation are based on the recent L^1-stability theory of Bressan, Liu, and Yang [8].

A posteriori error estimates are in fact specialized stability estimates. Let u and U be, respectively, exact and approximate solutions to equation (1.1.2). Define the residual $R(U)$ of the approximation U to be

$$U_t + f(U)_x = R(U). \tag{1.2.1}$$

$R(U)$ is computable a posteriori and is zero precisely when U is an exact solution. Subtracting (1.1.2) from (1.2.1) and writing $e = U - u$, we have the following equation for the error,

$$e_t + \widetilde{f}(e)_x = R(U), \tag{1.2.2}$$

where

$$\widetilde{f}(e) = \int_0^1 Df\big(\theta U + (1-\theta)(U-e)\big)\, d\theta.$$

In this context, a stability estimate for the non-homogeneous equation (1.2.2) is equivalent to an a posteriori error estimate for the approximation U.

The first residual-based a posteriori error estimate to be derived from Kuznetsovs' theory was the estimate of Cockburn and Gau [9]. In Chapter 3, we demonstrate that our estimate for scalar conservation laws in one space dimension coincides with their estimate. Although Cockburn has obtained other estimates for scalar conservation laws [10] and Hamilton-Jacobi equations [11], we will limit our discussion to the result below, since it is representative of his techniques.

Theorem 1.2.1 *Suppose that* $u : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ *is a piecewise smooth function which is the limit in* $L^1_{loc}(\mathbb{R}^d \times [0,T])$ *as* $\epsilon \to 0$ *of the solutions* $\{u_\epsilon\}$ *to the equation*

$$(u_\epsilon)_t + \nabla \cdot f(u_\epsilon) - \epsilon \Delta u_\epsilon = 0,$$

$$u_\epsilon(0,x) = u_0(x),$$

with f *smooth and* u_0 *piecewise smooth. Let* v *be any piecewise smooth function over* $\mathbb{R}^d \times [0,T]$ *with a finite number of discontinuities along smooth hyperplanes* $\{H_i\}$*, and satisfying*

$$\big|\big|u(t,\cdot) - v(t,\cdot)\big|\big|_{L^1(\mathbb{R}^d)} < \infty, \ \forall t.$$

Under these conditions

$$\begin{aligned} \big|\big|u(T) - v(T)\big|\big|_{L^1(\mathbb{R}^d)} \ &\leq \ \big|\big|u(0) - v(0)\big|\big|_{L^1(\mathbb{R}^d)} + \big|\big|v_t + \nabla \cdot f(v)\big|\big|_{L^1(\mathbb{R}^d \times [0,T])} \\ &\quad + \sum_i \big|\big|R_i(v)\big|\big|_{L^1(H_i)}, \end{aligned} \tag{1.2.3}$$

where

$$R_i(v) = \max\left\{0, \sup_{c\in[a,b]} \left\{ \Big(\big[F(v,c)\big]_{H_i}, \big[|v-c|\big]_{H_i} \Big) \cdot \vec{n}_{H_i} \right\}\right\}, \tag{1.2.4}$$

$[a, b]$ *is the range of the initial data,* $\vec{n}_{H_i}$ *is the normal with a negative time component to the surface* H_i*,* $F(v, c) = \operatorname{sign}(v - c)\big(f(v) - f(c)\big)$*, and*

$$\big[G(v)\big]_{H_i}(x) = \lim_{\epsilon\to 0} G\big(v(x+\epsilon\vec{n}_{H_i})\big) - G\big(v(x-\epsilon\vec{n}_{H_i})\big). \tag{1.2.5}$$

From the stability theory of Kuznetsov, Gosse and Makridakis [19] have also obtained estimates for E-schemes for 1-D scalar conservation laws. They were able to extend their results to high-order schemes satisfying certain weak entropy inequalities. The authors constructed adaptive schemes from their local estimates. Their estimates exhibited the correct qualitative behavior although quantitatively they tended to overestimate the error by a factor of 10^2-10^3.

Kröner and Ohlberger [29, 37] have published work together and separately on nonlinear scalar conservation laws and convection-diffusion equations in several space dimensions. Their results were developed for finite volume approximations and demonstrated with the techniques of Kuznetsov. Numerical experiments have confirmed their results and shown them to be accurate indicators of the local and global error.

The most productive techniques for the construction of a posteriori error estimates has been a combination of Galerkin orthogonality and of the stability for the dual problem. These tools have been restricted to the FEM and to conservation laws with viscosity. We make the distinction between estimates where the vanishing viscosity limit can be taken and those with constant non-vanishing viscosity. Among the latter, several estimates have been demonstrated for the Stokes, Oseen, and Navier-Stokes equations [1, 2, 35, 38, 44, 45].

The only published results for nonlinear systems of conservation laws with vanishing viscosity belong to Johnson and Szepessy [26]. Their estimates are an extension of Johnson's collaboration with Eriksson on parabolic equations [15]. Following [26], we give an informal statement of their estimate.

Consider $u : \mathbb{R} \times [0, T] \to \mathbb{R}^n$, a solution in H^2 of the system

$$u_t + f(u)_x - \epsilon u_{xx} = 0, \tag{1.2.6}$$

$$u(x,t) \to 0 \text{ as } |x| \to 0,$$

where f is smooth and strictly hyperbolic, and $u(\cdot, 0)$ is Lipschitz with compact support. Let U be an approximate solution constructed by the streamline-diffusion

FEM and let $\widehat{u}$ be the exact solution of (1.2.6) with ϵ replaced by

$$\widehat{\epsilon} = \max\left\{\epsilon, C_1 h\frac{|R(U)|}{|\nabla U|}, C_2 h^{3/2}\right\}, \tag{1.2.7}$$

where h is the mesh size and C_i are positive constants. Then

$$||\,\widehat{u} - U\,||_{L^2} \leq\; C^s C^i \big|\big|\widehat{\epsilon}^{-1} h^2 R(U)\big|\big|_{L^2}, \tag{1.2.8}$$

with C^i an interpolation constant, $C^s = C^s(\widehat{u}, U)$, and where integration occurs over space and time. This form of the estimate neglects the approximation error within the initial data and the time-discretization. The final estimate of the error between u and U would depend on an estimate of $u - \widehat{u}$ in terms of $\epsilon - \widehat{\epsilon}$.

We make a few remarks concerning estimate (1.2.8). First, we mention that the use of the L^2 norm is not appropriate for conservation laws since it is well-known that shock profiles are unstable in the L^p norms for $p > 1$. Secondly, from condition (1.2.7) one can see that the vanishing viscosity limit can only be achieved under mesh refinement. Moreover, their estimate is not technically a posteriori since the constant C^s depends on the exact solution $\widehat{u}$. They did propose several ways of computing C^s in practical applications but their only explicit bounds were obtained for non-interacting weak shock and rarefaction waves. It is because of these distinctions that the results of this dissertation can be said to be the first a posteriori error estimates for hyperbolic systems of nonlinear conservation laws.

Other residual-based error estimates in the vanishing viscosity limit were demonstrated by Süli and his collaborators [22, 42]. Their estimates apply to approximate solutions constructed by most FEM, including the discontinuous Galerkin FEM. They considered scalar linear hyperbolic equations and linear functionals of the solutions to such equations. Their results depended on Galerkin orthogonality and the stability of the dual problem.

In earlier work Süli, Houston, Mackenzie, and Warnecke [21] demonstrated a posteriori error estimates for the Friedrichs system occurring in linearized transonic flow. Their analysis was based on the stability theory of Friedrichs and therefore was applicable to a wide class of approximations, including the FEM and certain finite-difference methods. In particular, their method did not require Galerkin orthogonality or the introduction of artificial viscosity. They used the residual of the approximation to obtain two-sided a posteriori error bounds in the L^2 norm.

Nessayhu and Tadmor obtained one of the first a posteriori error estimates for scalar conservation laws in one space dimension. Although their result is stated in a dual Lip$^+$ norm, they nonetheless can obtain L^p estimates for convergence. The Lip$^+$ is essentially a one-sided Lipshitz norm measuring only the increasing variation. It is closely related to Oleinik's entropy condition.

This summary of the current state of the theory of a posteriori error estimation for hyperbolic equations emphasizes its relationship to stability theory. These theories all fail to generalize, in one way or another, to hyperbolic systems of nonlinear conservation laws. Methods based on Galerkin orthogonality and the stability of the dual problem remain limited to Sobolev norms and parabolic regularizations. Kuznetsov's and Tadmor's theory have not been extended to systems and Friedrichs theory is restricted to smooth solutions. The recent stability theories of Bressan [12], Liu and Yang [34], and Hu and LeFloch [23] therefore offer an exciting opportunity for the construction of new a posteriori error estimates for systems of conservation laws. In this dissertation, we obtain a posteriori error estimators from the work of Bressan, Liu, and Yang [8].

Chapter 2

Front-Tracking and the Liu-Yang Functional

This chapter discusses preliminaries for the error estimates of chapters 3 and 4. The error estimates for those chapters apply to approximations constructed by the front-tracking method as described in Section 2.1. The main tool in the proof of the a posteriori error estimate is the functional of Liu and Yang. This functional is equivalent to the L^1 norm and will be used to measure the distance between approximations. In the last section, we define the discrepancy which is used to measure the error in our front-tracking approximations.

2.1 The Front-Tracking Method

We now describe the front-tracking method, in the spirit of Risebro [41], and present some fundamental a priori estimates. Dafermos [14] was the first to consider piecewise constant approximate solutions to conservation laws, albeit for scalar equations. Di Perna [39] then extended the method to 2×2 systems and later, Bressan [6] and Risebro [41] independently proposed constructions for $n \times n$ systems. Some have also implemented this method as a numerical scheme [20, 32]. The front-tracking method is intimately related to Glimm's scheme [17] and to Liu's wave-tracing extension [33].

We mention that there exists an entirely different method also going by the name of front-tracking. This front-tracking method, initially advocated by Richtmeyer and von Neumann [46], is currently being implemented by several groups and most notably by Glimm and his associates [18]. In this work, front-tracking will refer exclusively to the method of Dafermos and its extensions.

In front-tracking, the key point in the construction of the piecewise constant approximations is to control the total number of discontinuities so that the approximation may be extended to all times. We present Risebro's version of the front-tracking algorithm, omitting Bressan's [5, 6]. We begin with a few preliminary definitions which will help to clarify the construction. For convenience, given a function u of time t and space x let $u(t)$ denote the function of space $u(t)(x) = u(t,x)$.

Definition 2.1.1 *A wave in a piecewise constant function* $u(t) : \mathbb{R} \to \Omega$ *is a pair* $(z,k) \in \mathbb{R} \times \{1,\dots,n\}$ *such that the states* u^- *and* u^+ *separating the discontinuity at* $x = z$ *satisfy* $p_k(u^-,u^+) \neq 0$. *If* $\alpha = (z,k)$ *is a wave then we say that* α *belongs to the discontinuity at* z *and let*

i) $k_\alpha \equiv k$ *denote its characteristic type,*

ii) $\sigma_\alpha \equiv p_k(u^-,u^+)$ *denote its strength, and*

iii) $x_\alpha \equiv z$ *denote its position.*

The set of all waves in $u(t)$ *will be denoted by* $\mathcal{W}\big(u(t)\big)$.

Definition 2.1.2 *A wave* α *is called isolated if it is the only wave located at the position* x_α. *If the wave* α *belongs to a genuinely nonlinear family* k_α, *then it is called a shock wave if* $\sigma_\alpha < 0$ *and a rarefaction wave otherwise. If the family* k_α *is linearly degenerate, then it is called a linearly degenerate wave.*

Let $\mathcal{D}\big(u(t)\big)$ *denote the set of points in space where the approximation* $u(t)$ *has discontinuities.*

We note that for an isolated wave α, the left and right sided limits of u at x_α, denoted u^- and u^+, are related by

$$u^+ = T_{k_\alpha}(\sigma_\alpha)(u^-).$$

Definition 2.1.3 *Let* $\alpha = (z,k) \in \mathcal{W}\big(u(t)\big)$ *be a wave, belonging to a discontinuity at* $x = z$, *with neighboring states* u^- *and* u^+. *We define the left and right hand states of* α *to be*

$$u_\alpha^- = T_{k-1}\big(p_{k-1}(u^-,u^+)\big) \circ \cdots \circ T_1\big(p_1(u^-,u^+)\big)(u^-), \text{ and} \qquad (2.1.1)$$

$$u_\alpha^+ = T_k(\sigma_\alpha)(u_\alpha^-).$$

Definition 2.1.4 *We say that a wave* α *is dominant if it travels at a speed*

$$\dot{x}_\alpha \in \lambda_{k_\alpha}(\Omega). \qquad (2.1.2)$$

We now construct a front-tracking approximation for (1.1.2) with initial data $\bar{u}$ of bounded variation. Choose $u(0,\cdot)$ a piecewise constant function with a finite number of discontinuities and such that $||\bar{u} - u(0,\cdot)||_{L^1}$ is finite. Suppose that each discontinuity in the approximation $u(0,\cdot)$ separated by states u^- and u^+, is of the form

$$u^+ = T_k\big(p_k(u^-, u^+)\big)(u^-). \tag{2.1.3}$$

If we propagate one of these initial discontinuities forward in time along a smooth trajectory $z(t)$ then we will assume that

$$\dot{z} \in \lambda_k(\Omega) \tag{2.1.4}$$

for the integer k in (2.1.3). Therefore initially, each discontinuity corresponds to exactly one isolated wave.

When two discontinuities meet, we use an approximate Riemann solver in the class of piecewise constant functions. Suppose that two discontinuities, separated by three consecutive states u_l, u_m, and u_r meet at time t_1. Let $u_0 = u_l, u_1, \ldots, u_n = u_r$ be the $n+1$ states of Theorem 1.1.1 for the Riemann problem at time t_1. In a neighborhood of the collision, construct a piecewise constant function composed of n regions separated by the states $u_0, \ldots, u_n$ and such that

i) if $u_k = S_k\big(p_k(u_l, u_r)\big)(u_{k-1})$ then these states are separated by an isolated wave α traveling at a speed

$$\dot{x}_\alpha \in \lambda_k(\Omega); \tag{2.1.5}$$

ii) if $u_k = R_k\big(p_k(u_l, u_r)\big)(u_{k-1})$ then there exists a finite number of isolated rarefaction waves $\alpha(1), \ldots, \alpha(j)$ satisfying

$$p_k(u_l, u_r) = \sum_{i=1}^{j} \sigma_{\alpha(i)}, \tag{2.1.6}$$

$$\dot{x}_{\alpha(i-1)} < \dot{x}_{\alpha(i)}, \tag{2.1.7}$$

and

$$\dot{x}_{\alpha(i)} \in \lambda_k(\Omega), \forall i. \tag{2.1.8}$$

For consistency, we also require that neighboring isolated shock waves not collide.

There is clearly more than one possible solution which satisfies these conditions. In particular, there is freedom in the choice of the speeds $\dot{x}_\alpha$, and in the number and strength of the rarefaction waves $\alpha(1), \ldots, \alpha(j)$.

Using the approximate Riemann solver described above two different situations may occur. Either the approximation will blow-up in finite time, in the L^1, L^∞, or total variation norm, or the approximation will develop an infinite number of discontinuities in finite time. Both of these difficulties require us to examine the global interaction patterns within the approximation. We say that two waves $\alpha, \beta \in \mathcal{W}\big(u(t)\big)$ are *approaching* if $x_\alpha < x_\beta$ and $k_\alpha > k_\beta$, or if $k_\alpha = k_\beta$ and at least one of these waves is a shock. Define the interaction potential to be

$$Q\big(u(t)\big) = \sum_{\alpha \text{ approaches } \beta} |\sigma_\alpha \cdot \sigma_\beta|, \tag{2.1.9}$$

and consider the following functional

$$V(u(t)) = \sum_{\alpha \in \mathcal{W}\left(u(t)\right)} |\sigma_\alpha|, \tag{2.1.10}$$

equivalent to the total variation norm $||u(t)||_{\mathrm{TV}}$. Let t_k denote the time at which the k-th interaction occurred and define

$$P_k = \sum_{\alpha,\beta} |\sigma_\alpha \cdot \sigma_\beta| \tag{2.1.11}$$

where the sum occurs over pairs of waves which approach and collide at time t_k.

Following the argument of Glimm [17], Risebro showed that if the total variation of the initial data is smaller than some a priori bound then there exists a constant K such that the functional

$$V\big(u(t)\big) + KQ\big(u(t)\big) \tag{2.1.12}$$

is decreasing in time. This implies that the approximations remain total variation bounded uniformly in time and therefore also bounded in L^∞. This argument also shows that for a fixed positive Γ_u, there exists an integer N_1 such that

$$P_k < \Gamma_u, \forall k \geq N_1. \tag{2.1.13}$$

Fix the constant Γ_u and let t_{N_1} be the time satisfying (2.1.13) at which two isolated waves collide, say α and β with $k_\alpha < k_\beta$. Let $u_0, u_1, \ldots, u_n$ be the $n+1$ states of Theorem 1.1.1 for the Riemann problem at the collision of α and β. Instead

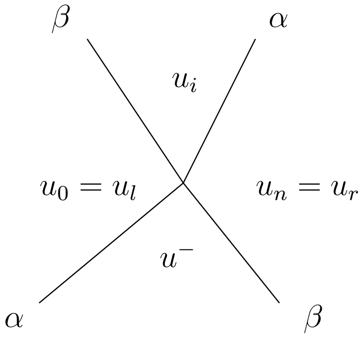

Figure 2.1: The interaction of the discontinuities α and β when condition (2.1.13) is satisfied.

of several outgoing isolated waves, we regroup these waves into two discontinuities traveling at the speeds of the k_α and k_β families. Choose the state to the right of the outgoing α-discontinuity to be u_n and choose the state to the left of the outgoing β-discontinuity to be u_0. Pick any integer i, satisfying $k_\alpha < i \leq k_\beta$, and let the state between both discontinuities be u_i. If there were several colliding pairs of discontinuities at time t_{N_1}, the same operation would apply to each pair. More generally, we now require that neighboring dominant shock waves not collide. This construction is illustrated in Figure 2.1.

Having constructed a new approximation u for the times immediately following t_{N_1}, we return to our earlier approximate Riemann solver to resolve all new interactions at times after t_{N_1}. When a time t_{N_2} is reached such that $P_k < \Gamma_u$, for all $k \geq N_2$, then construct the approximation using non-isolated waves.

A time will be reached when (2.1.13) is always satisfied and no new discontinuities are created. Since the system is strictly hyperbolic, eventually all discontinuities will cross paths and the approximation will consist of a finite number of discontinuities moving apart. The approximation will therefore be defined for all times. This method satisfies the CFL condition

$$|\dot{x}_\alpha| \leq \Lambda, \, \forall \alpha \in \mathcal{W}\big(u(t)\big), \tag{2.1.14}$$

for the constant Λ defined in (1.1.23). We summarize the result of Risebro.

Lemma 2.1.1 *For any Γ_u, there exists positive constants δ and K, depending on the system* (1.1.2) *but independent of Γ_u, such that any front-tracking approximation u satisfying*

$$||u(0)||_{L^\infty} + ||u(0)||_{\mathrm{TV}} < \delta, \tag{2.1.15}$$

is defined for all t and has values in Ω. Moreover, for this approximation the functional

$$V\big(u(t)\big) + KQ\big(u(t)\big), \tag{2.1.16}$$

is decreasing in time.

The algorithm outlined above does not generate a unique front-tracking approximation and in fact when Γ_u is large the approximation may be very different from the exact solution. In the original construction of Risebro, a narrower class of approximations was considered so that existence results for the initial value problem (1.1.2) could be achieved. Each of the approximations of Risebro depended on a small parameter ϵ and therefore will be called an ϵ-approximation.

In the construction of Risebro, each ϵ-approximation u for equation (1.1.2) with initial data $\bar{u} \in L^1(\mathbb{R}, \mathbb{R}^n)$ satisfied the following conditions:

i) for any dominant wave $\alpha = (z, k) \in \mathcal{W}\big(u(t)\big)$,

$$\left|\dot{x}_\alpha - s_k(u_\alpha^-, u_\alpha^+)\right| < \epsilon,$$

ii) for any isolated rarefaction wave $\alpha \in \mathcal{W}\big(u(t)\big)$, $\sigma_\alpha < \epsilon$,

iii) $\Gamma_u < \epsilon$, and

iv) $||u(0) - \bar{u}||_{L^1} < \epsilon$.

Under these conditions, he showed the following.

Theorem 2.1.1 *[41] Given the system (1.1.2) there exists a constant $\delta_0 > 0$ such that for any initial data $\bar{u}$ satisfying $||\bar{u}||_{L^\infty} + ||\bar{u}||_{\mathrm{TV}} < \delta_0$, there exists a sequence $\{u_\epsilon\}$ of ϵ-approximations which converge in $L^1_{loc}([0,\infty) \times \mathbb{R})$ as $\epsilon \to 0$ to a weak solution of the system with initial data $\bar{u}$. If there exists an entropy-entropy flux pair for the system then the limit is entropy admissible.*

2.2 The Liu-Yang Functional

Following [8], we construct the functional Φ of Liu and Yang. Let u and v be two front-tracking approximations and assume that $u(0) - v(0) \in L^1$. Since the total variation of both solutions remains bounded, $u(t) - v(t) \in L^1$ for all times t. The distance between both approximate solutions $u(t)$ and $v(t)$ will be measured in the shock coordinates of Theorem 1.1.2. Since the shock coordinates will only be used for this purpose and the time t will be fixed throughout, we will abbreviate

$$q_k\big(u(t,x), v(t,x)\big) = q_k(x).$$

Thus $q_i(x) < 0$ indicates that to reach v from u, a shock wave of the i-th family is used while for $q_i(x) > 0$, a rarefaction shock is used.

The functional of Liu and Yang

$$\Phi\big(u(t), v(t)\big) = \sum_{i=1}^{n} \int_{-\infty}^{\infty} |q_i(x)| W_i(x)\, dx \tag{2.2.1}$$

is finite if the W_i are uniformly bounded. It is convenient to require

$$1 \leq W_i(x) \leq 2,\ \forall x \in \mathbb{R}, \tag{2.2.2}$$

uniformly in time. Let

$$W_i(x) = 1 + \kappa_1 \Big(A_i(x) + B_i(x) \Big) + \kappa_2 \Big(Q\big(u(t)\big) + Q\big(v(t)\big) \Big) \tag{2.2.3}$$

where the terms A_i and B_i are given below and the constants κ_1, κ_2 will be chosen to satisfy (2.2.2). Let

$$A_i(x) = \Big(\sum_{\substack{\alpha \in \mathcal{W}(u) \bigcup \mathcal{W}(v) \\ x_\alpha < x,\, k_\alpha > i}} + \sum_{\substack{\alpha \in \mathcal{W}(u) \bigcup \mathcal{W}(v) \\ x_\alpha > x,\, k_\alpha < i}} \Big) |\sigma_\alpha| \tag{2.2.4}$$

measure the strength of all waves α, $k_\alpha \neq i$, with the potential to interact with the virtual wave q_i. When the i-th family is linearly degenerate, $B_i \equiv 0$. If the i-th family is genuinely nonlinear then let

$$B_i(x) = \begin{cases} \Big(\sum_{\substack{\alpha \in \mathcal{W}(u) \\ x_\alpha < x,\, k_\alpha = i}} + \sum_{\substack{\alpha \in \mathcal{W}(v) \\ x_\alpha > x,\, k_\alpha = i}} \Big) |\sigma_\alpha| & \text{if } q_i(x) < 0, \\ \Big(\sum_{\substack{\alpha \in \mathcal{W}(v) \\ x_\alpha < x,\, k_\alpha = i}} + \sum_{\substack{\alpha \in \mathcal{W}(u) \\ x_\alpha > x,\, k_\alpha = i}} \Big) |\sigma_\alpha| & \text{if } q_i(x) > 0, \end{cases} \tag{2.2.5}$$

measure the strength of the waves of the family $k_\alpha = i$ with the potential to interact with q_i. We will return to study B_i after the next lemma. Many of our later arguments will be localized around a specific wave α and it will therefore be natural, when it is clear from the context, to suppress the symbol α. For a quantity G, we will then write

$$G^{\pm} = \lim_{x \to x_\alpha \pm} G(x).$$

We also introduce the notation $[G]_\alpha = G^+ - G^-$ to denote the jump in a quantity G across a wave located at x_α.

Lemma 2.2.1 *If $\alpha \in \mathcal{W}\big(v(t)\big)$ is an isolated wave then for $i \neq k_\alpha$,*

$$[W_i]_\alpha = \begin{cases} \kappa_1|\sigma_\alpha| & \text{if } i < k_\alpha, \\ -\kappa_1|\sigma_\alpha| & \text{if } i > k_\alpha. \end{cases} \tag{2.2.6}$$

If α is linearly degenerate then $[W_{k_\alpha}]_\alpha = 0$ and otherwise

$$[W_{k_\alpha}]_\alpha = \begin{cases} \kappa_1|\sigma_\alpha| & \text{if } q^-_{k_\alpha} \text{ and } q^+_{k_\alpha} \text{ are both positive}, \\ -\kappa_1|\sigma_\alpha| & \text{if } q^-_{k_\alpha} \text{ and } q^+_{k_\alpha} \text{ are both negative}, \\ \kappa_1[B_{k_\alpha}]_\alpha & \text{otherwise}. \end{cases} \tag{2.2.7}$$

Suppose that α is a genuinely nonlinear wave. Then if $q^+_{k_\alpha} < 0 < q^-_{k_\alpha}$,

$$B^-_{k_\alpha} + B^+_{k_\alpha} + |\sigma_\alpha| = \Big(\sum_{\substack{\beta\in\mathcal{W}(u) \\ k_\beta = k_\alpha}} + \sum_{\substack{\beta\in\mathcal{W}(v) \\ k_\beta = k_\alpha}} \Big)|\sigma_\beta|, \tag{2.2.8}$$

and if $q^-_{k_\alpha} < 0 < q^+_{k_\alpha}$,

$$B^-_{k_\alpha} + B^+_{k_\alpha} = |\sigma_\alpha| + \Big(\sum_{\substack{\beta\in\mathcal{W}(u) \\ k_\beta = k_\alpha}} + \sum_{\substack{\beta\in\mathcal{W}(v) \\ k_\beta = k_\alpha}} \Big)|\sigma_\beta|. \tag{2.2.9}$$

Proof We first consider identity (2.2.6). It is easy to verify that

$$[W_i]_\alpha = \kappa_1\Big([A_i]_\alpha + [B_i]_\alpha\Big). \tag{2.2.10}$$

For the first term $[A_i]_\alpha$, it suffices to consider in equation (2.2.4) only waves belonging to $\mathcal{W}\big(u(t)\big)$. If $i < k_\alpha$, the wave α appears in neither sum of A^-_i but does appear in the first sum of A^+_i. We therefore have $\kappa_1[A_i]_\alpha = \kappa_1|\sigma_\alpha|$. Similarly, when $i > k_\alpha$ the wave α belongs to the second term in A^-_i but in neither of the sums in A^+_i. In this case $\kappa_1[A_i]_\alpha = -\kappa_1|\sigma_\alpha|$. To verify that $[B_i]_\alpha = 0$, we notice that across a discontinuity of the k_α-th family the quantity q_i can only change by a second-order quantity. Since q_i is first order, it is greater than this change. This implies that the same sums appear in both B^-_i and B^+_i.

We now turn to (2.2.7). The jump $[A_{k_\alpha}]_\alpha = 0$ since it only involves waves of families other than k_α. If α is linearly degenerate then $B_{k_\alpha} = 0$ and (2.2.10) shows that $[W_{k_\alpha}]_\alpha = 0$. We now assume that α is genuinely nonlinear and that both $q^-_{k_\alpha}$ and $q^+_{k_\alpha}$ are positive. In this case, the term $B^+_{k_\alpha}$ contains a sum over all k_α-waves in v located to left of x_α, including α. $B^-_{k_\alpha}$ has the same sum but without the wave α. The jump across α is therefore $[W_{k_\alpha}]_\alpha = \kappa_1[B_{k_\alpha}]_\alpha = \kappa_1|\sigma_\alpha|$. The case when both $q^\pm_{k_\alpha}$ are negative is similar. When the sign of q_{k_α} changes across the wave α, the waves in the sums $B^{\pm}_{k_\alpha}$ are distinct, except possibly for the occurrence of α. Therefore, there is no cancellation between their difference and we measure this jump to be $\kappa_1[B_{k_\alpha}]_\alpha$.

For the demonstration of (2.2.8), assume that α is a genuinely nonlinear wave and that $q^+_{k_\alpha} < 0 < q^-_{k_\alpha}$. From the definition (2.2.5) of $B_{k_\alpha}(x)$, in a neighborhood of $\dot{x}_\alpha$ we have

$$B^-_{k_\alpha} = \Big(\sum_{\substack{\beta\in\mathcal{W}(v)\\ x_\beta<x_\alpha,k_\beta=k_\alpha}} + \sum_{\substack{\beta\in\mathcal{W}(u)\\ x_\beta\geq x_\alpha,k_\beta=k_\alpha}} \Big)|\sigma_\beta|, \tag{2.2.11}$$

$$B^+_{k_\alpha} = \Big(\sum_{\substack{\beta\in\mathcal{W}(u)\\ x_\beta\leq x_\alpha,k_\beta=k_\alpha}} + \sum_{\substack{\beta\in\mathcal{W}(v)\\ x_\beta> x_\alpha,k_\beta=k_\alpha}} \Big)|\sigma_\beta|, \tag{2.2.12}$$

where the sums occur over disjoint sets of waves, none of which include the shock wave α. To prove (2.2.8), it suffices to sum (2.2.11) and (2.2.12).

Definition 2.2.5 states that for $q^-_{k_\alpha} < 0 < q^+_{k_\alpha}$,

$$B^-_{k_\alpha} = \sum_{\substack{\beta\in\mathcal{W}(u)\\ x_\beta<x_\alpha,\, k_\beta=k_\alpha}} |\sigma_\beta| + \sum_{\substack{\beta\in\mathcal{W}(v)\\ x_\beta\geq x_\alpha,\, k_\beta=k_\alpha}} |\sigma_\beta|, \tag{2.2.13}$$

$$B^+_{k_\alpha} = \sum_{\substack{\beta\in\mathcal{W}(v)\\ x_\beta\leq x_\alpha,\, k_\beta=k_\alpha}} |\sigma_\beta| + \sum_{\substack{\beta\in\mathcal{W}(u)\\ x_\beta> x_\alpha,\, k_\beta=k_\alpha}} |\sigma_\beta|. \tag{2.2.14}$$

Notice that the wave α occurs in both the second sum of (2.2.13) and the first sum of (2.2.14). Besides the wave α appearing twice, all other waves of the k_α family in u and v appear exactly once in either $B^-_{k_\alpha}$ or $B^+_{k_\alpha}$. This demonstrate (2.2.9). ■

We now support the claim that $B_i(x)$ represents the sum of the strength of the waves α of the i-th family, $i \neq k_\alpha$, which have the potential to interact with q_i. For definiteness, choose a rarefaction wave $\alpha \in \mathcal{W}\big(v(t)\big)$ for which $0 \leq q^-_{k_\alpha} \leq q^+_{k_\alpha}$. This situation is described in Figure 2.2. For each integer k, let $w_k(x)$ be the intermediate states defined by

$$w_k(x) = S_k\big(q_k(x)\big) \circ \cdots \circ S_1\big(q_1(x)\big)\big(u(t,x)\big). \tag{2.2.15}$$

If the k_α-th shock of strength q_{k_α} joining u and v is interpreted as a virtual wave, then its virtual speed can be defined as $s_{k_\alpha}\big(w_{k_\alpha}(x), w_{k_\alpha-1}(x)\big)$. Following this analogy, along the k_α-th shock coordinate the wave α has a virtual speed $s_{k_\alpha}(w^-_{k_\alpha}, w^+_{k_\alpha})$, faster than the speeds of the virtual waves $q^-_{k_\alpha}$ and $q^+_{k_\alpha}$. More specifically, the virtual speeds

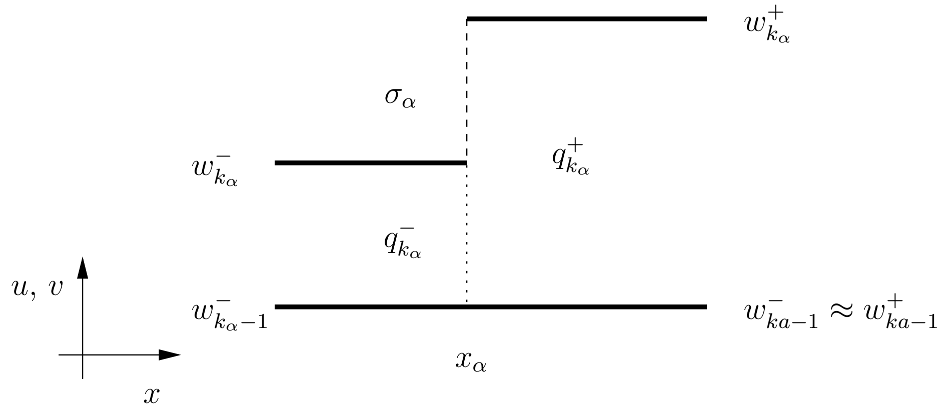

Figure 2.2: Figure illustrating the potential interaction between α and the virtual waves $q^-_{k_\alpha}$ and $q^+_{k_\alpha}$. The bold curve represents the graph of u and v in the k_α-th shock coordinate.

of $q^-_{k_\alpha}$, $q^+_{k_\alpha}$, and α are respectively

$$s_{k_\alpha}(w^-_{k_\alpha-1}, w^-_{k_\alpha}) < s_{k_\alpha}(w^+_{k_\alpha-1}, w^+_{k_\alpha}) < s_{k_\alpha}(w^-_{k_\alpha}, w^+_{k_\alpha}). \tag{2.2.16}$$

Therefore, as long as $q_{k_\alpha}(x)$ remains positive the waves to the left of x will approach it. The positivity of $[W_{k_\alpha}]_\alpha$ is essentially a consequence of the inequalities (2.2.16). Similarly, when $q^-_{k_\alpha} \leq q^+_{k_\alpha} \leq 0$ the virtual speed of α is slower than those of either $q^-_{k_\alpha}$ and $q^+_{k_\alpha}$,

$$s_{k_\alpha}(w^-_{k_\alpha}, w^+_{k_\alpha}) < s_{k_\alpha}(w^-_{k_\alpha-1}, w^-_{k_\alpha}) < s_{k_\alpha}(w^+_{k_\alpha-1}, w^+_{k_\alpha}). \tag{2.2.17}$$

In other words, only the waves to the right of x approach q_{k_α}. In this case, (2.2.7) predicts a negative contribution to $[W_{k_\alpha}]_\alpha$.

Assuming that κ_1 is some known but large constant, the next lemma states that condition (2.2.2) can be satisfied. The value of κ_1 will be determined later in the proof.

Lemma 2.2.2 *For any fixed value of κ_1, there exists δ_0 and κ_2, depending only on κ_1 and the system* (1.1.2)*, such that any front-tracking approximations u and v satisfying*

$$||u(0)||_{L^\infty} + ||u(0)||_{\mathrm{TV}},\ ||v(0)||_{L^\infty} + ||v(0)||_{\mathrm{TV}} < \delta_0, \tag{2.2.18}$$

$$u(0) - v(0) \in L^1(\mathbb{R}, \Omega),$$

are defined for all t and have values in Ω. For these approximations, the functional Φ exists, condition (2.2.2) *holds, and the weights W_i decrease in time.*

Proof With the constants δ and K of Lemma 2.1.1, take any front-tracking approximation u satisfying

$$||u(0)||_{L^\infty} + ||u(0)||_{\mathrm{TV}} < \delta. \tag{2.2.19}$$

Lemma 2.1.1 assures us that u is defined for all t, takes values in Ω, and that the functional

$$V\big(u(t)\big) + KQ\big(u(t)\big) \tag{2.2.20}$$

decreases in time. Remark that $Q(\cdot) \leq V(\cdot)^2$ and that there exists a constant C such that

$$V(\cdot) \leq C|| \cdot ||_{\mathrm{TV}}. \tag{2.2.21}$$

We assume that

$$\delta_0 \leq \min\left\{\delta, \frac{1}{4C\kappa_1}, \frac{1}{CK}\right\}. \tag{2.2.22}$$

and that the approximations u and v satisfy (2.2.18). Under these conditions, we have

$$\begin{aligned} V\big(u(t)\big) + KQ\big(u(t)\big) &\leq V\big(u(0)\big) + KQ\big(u(0)\big) \\ &\leq C||u(0)||_{\mathrm{TV}} + KC^2||u(0)||^2_{\mathrm{TV}} \\ &\leq 2C||u(0)||_{\mathrm{TV}} \leq \frac{1}{2\kappa_1}. \end{aligned} \tag{2.2.23}$$

An identical estimate also holds for the approximation v. In conclusion, if u and v both satisfy conditions (2.2.19) and (2.2.22), then summing inequalities (2.2.23) for u and v we find

$$\Big(V\big(u(t)\big) + V\big(v(t)\big)\Big) + K\Big(Q\big(u(t)\big) + Q\big(v(t)\big)\Big) \leq \frac{1}{\kappa_1}, \forall t. \tag{2.2.24}$$

Since the sums in A_i and B_i occur over subsets of those in $V\big(u(t)\big) + V\big(v(t)\big)$, the functional

$$\Big(A_i(x) + B_i(x)\Big) + K\Big(Q\big(u(t)\big) + Q\big(v(t)\big)\Big) \tag{2.2.25}$$

is also uniformly bounded in time by $1/\kappa_1$. Moreover, since the decay property of $V + KQ$ is in fact a consequence of the monotonicity of the interaction potential Q and of the smallness of V, the functional (2.2.25) also inherits this property.

Multiplying (2.2.24) by κ_1, defining $\kappa_2 = \kappa_1 K$, and adding one, we notice that condition (2.2.2) is satisfied. Since the approximations are total variation bounded for all time, the difference $u(t) - v(t)$ remains in L^1 for all time and the functional Φ is finite. ■

Definition 2.2.1 *Any front-tracking approximation satisfying conditions* (2.2.18) *of Lemma 2.2.2 will be called admissible.*

At times when no interactions occur, the functional Φ is smooth in time and the functions $W_i(x)$ are constant in the regions which do not intersect the trajectories of waves. On the other hand, at a discrete time t_0 when an interaction does occur at $x_0 \in \mathbb{R}$, Lemma 2.2.2 implies that

$$\lim_{t \to t_0+} W_i(x) \leq \lim_{t \to t_0-} W_i(x), \qquad x \neq x_0. \tag{2.2.26}$$

Since the terms W_i decrease during interactions, so does the functional Φ. The main step will therefore be an analysis of Φ during those times when it is smooth.

2.3 Residuals and Discrepancies

The residual $R(v) = v_t + f(v)_x$ of an approximation v is not a sufficient measure of the error for weak solutions to hyperbolic conservation laws. We therefore propose a closely related measure which can also be computed from the approximate solution, the initial data, and the flux.

The primary difficulty with the residual $R(v)$ is that it cannot distinguish between an unphysical rarefaction shock solution and a physical rarefaction fan. Moreover, it is technically inconvenient as it measures the error in phase-space coordinates while the functional Φ measures distance along shock coordinates. To resolve these difficulties, we introduce the notion of discrepancies for waves.

We begin by observing that for a smooth function V of x and t the residual coincides with the integral form of the conservation law

$$\begin{aligned}\int_{-l}^{l} R(V)\ dx &= \int_{-l}^{l} V_t + f(V)_x\ dx \\ &= \frac{d}{dt}\int_{-l}^{l} V\ dx + f(V)\Big|_{-l}^{l}.\end{aligned} \tag{2.3.1}$$

Since the differential expression $V_t + f(V)_x$ for conservation laws is derived from the integral form of the conservation law, we will assume that the residual for non-smooth functions is also given by identity (2.3.1). Consider now an isolated discontinuity located at $z(t)$ in the approximation v, with neighboring states v^- and v^+, and which for convenience we assume is initially located at the origin $z(0) = 0$. We now construct a discontinuity traveling with the $t = 0$ velocity of the discontinuity, namely

$$V(t,x) = \begin{cases} v^- & \text{for } x < \dot{z}(0)t, \\ v^+ & \text{for } x \geq \dot{z}(0)t. \end{cases}$$

Let V_r be the Riemann solution of (1.1.2) with initial data $V(0,\cdot)$. For any $l > 0$, the weak solution V_r satisfies the integral form of the conservation law

$$\int_{-l}^{l} R(V_r)\ dx = \frac{d}{dt}\int_{-l}^{l} V_r(t,x)\ dx - f\big(V_r(t,\cdot)\big)\Big|_{-l}^{l} = 0, \tag{2.3.2}$$

while in general

$$\int_{-l}^{l} R(V)\ dx = \frac{d}{dt}\int_{-l}^{l} V(t,x)\ dx - f\big(V(t,\cdot)\big)\Big|_{-l}^{l} \neq 0. \tag{2.3.3}$$

If l is sufficiently large then for $t > 0$ the fluxes $f(V)$ and $f(V_r)$ agree at the boundaries of $[-l, l]$. Subtracting (2.3.2) from (2.3.3), extending the domain of integration to the real line, and then performing a change of variable into self-similar coordinates $\xi = x/t$, we obtain

$$\begin{aligned}\int_{-l}^{l} R(V)\ dx &= \frac{d}{dt}\int_{-l}^{l} V(t,x)\ dx - f\big(V(t,\cdot)\big)\Big|_{-l}^{l} \\ &= \frac{d}{dt}\int_{-l}^{l} \Big(V(t,x) - V_r(t,x)\Big)\ dx \\ &= \frac{d}{dt}\int_{-\infty}^{\infty} \Big(V(\xi) - V_r(\xi)\Big)\ t\ d\xi = \int_{-\infty}^{\infty} \Big(V(\xi) - V_r(\xi)\Big)\ d\xi.\end{aligned} \tag{2.3.4}$$

The absolute value of the residual $|\int R\ dx|$ is clearly zero for isolated rarefaction waves in front-tracking approximations. For a discontinuity at an arbitrary position

$z \in \mathbb{R}$, by a change of variable, the discontinuity can be centered at the origin and the self-similar functions V and V_r can be constructed from the knowledge of the states of v neighboring the origin. The following can therefore be computed entirely from the approximation v, i.e. the quantity is computable a posteriori.

Definition 2.3.1 *Define the phase-space discrepancy of the discontinuity located at z at time t, to be*

$$D_z^*(t) = \int_{-\infty}^{\infty} ||V(\xi) - V_r(\xi)||\, d\xi, \tag{2.3.5}$$

where $||\cdot||$ is the euclidean norm in phase-space.

To measure the discrepancy in wave coordinates, introduce the coordinates $\widetilde{p}_k(\xi)$ satisfying

$$\begin{array}{rcl} V_r(\xi) & = & T_n\big(\widetilde{p}_n(\xi)\big) \circ \cdots \circ T_1\big(\widetilde{p}_1(\xi)\big)\big(V(\xi)\big), \text{ for } \xi < \dot{z}(0), \\ V(\xi) & = & T_n\big(\widetilde{p}_n(\xi)\big) \circ \cdots \circ T_1\big(\widetilde{p}_1(\xi)\big)\big(V_r(\xi)\big), \text{ for } \xi \geq \dot{z}(0). \end{array} \tag{2.3.6}$$

To treat shock waves consistently, their strengths must be measured along the shock curves S_k both before and after the point $\xi = \cot z$. This explains why the order in which we measure the distance between V_r and V must be reversed as we cross the line $\xi = \dot{z}$. This procedure also identifies the components of the discontinuity belonging to each underlying wave.

Definition 2.3.2 *Define the discrepancy of the wave $\alpha = (z, k)$ to be*

$$D_\alpha(t) = \int_{-\infty}^{\infty} |\widetilde{p}_k(\xi)|\, d\xi. \tag{2.3.7}$$

The topology defined by the phase-space and wave coordinates are equivalent. More precisely, there exists constants c_1 and c_2 such that for any two states, u and v, we have

$$\begin{array}{rcccl} c_1||u-v|| & \leq & \displaystyle\sum_{k=1}^{n} |p_k(u,v)| & \leq & c_2||u-v||, \\ c_1||u-v|| & \leq & \displaystyle\sum_{k=1}^{n} |q_k(u,v)| & \leq & c_2||u-v||. \end{array} \tag{2.3.8}$$

These bounds imply that for any discontinuity at $z \in \mathbb{R}$,

$$c_1 D_z^*(t) \leq \sum_{k=1}^{n} D_{(z,k)}(t) \leq c_2 D_z^*(t). \tag{2.3.9}$$

In fact, the integrand $\widetilde{p}_k$ in (2.3.7) can be computed as if α was an isolated wave. If v_α^- and v_α^+ are respectively the left and right hand states of the wave $\alpha = (z, k)$, as given in Definition 2.1.3, then let

$$\widetilde{V}(t,x) = \begin{cases} v_\alpha^- & \text{for } x < \dot{z}(0)t, \\ v_\alpha^+ & \text{for } x \geq \dot{z}(0)t, \end{cases}$$

and let $\widetilde{V}_r$ be the solution to the Riemann problem with initial data $\widetilde{V}(0,\cdot)$. It is now easy to verify that the integrand $\widetilde{p}_k$, originally introduced in 2.3.6, also satisfies

$$\begin{aligned} \widetilde{V}_r(\xi) &= T_k\big(\widetilde{p}_k(\xi)\big)\big(\widetilde{V}(\xi)\big), \text{ for } \xi < \dot{z}(0), \\ \widetilde{V}(\xi) &= T_k\big(\widetilde{p}_k(\xi)\big)\big(\widetilde{V}_r(\xi)\big), \text{ for } \xi \geq \dot{z}(0). \end{aligned} \tag{2.3.10}$$

Lemma 2.3.1 *If $\alpha \in \mathcal{W}(v(t))$ is a shock wave or a linearly degenerate wave then*

$$D_\alpha(t) = |\sigma_\alpha|\big|\dot{x}_\alpha - s_{k_\alpha}(v_\alpha^-, v_\alpha^+)\big|. \tag{2.3.11}$$

If α is a rarefaction wave then

$$D_\alpha(t) = \begin{cases} \frac{1}{2}\big(\lambda_{k_\alpha}(v_\alpha^-) - \dot{x}_\alpha\big)^2 + \frac{1}{2}\big(\dot{x}_\alpha - \lambda_{k_\alpha}(v_\alpha^+)\big)^2, & \textit{if } \dot{x}_\alpha \in [\lambda_{k_\alpha}(v_\alpha^-), \lambda_{k_\alpha}(v_\alpha^+)] \\ |\sigma_\alpha|\big|\dot{x}_\alpha - \frac{1}{2}\big(\lambda_{k_\alpha}(v_\alpha^-) + \lambda_{k_\alpha}(v_\alpha^+)\big)\big|, & \textit{otherwise.} \end{cases} \tag{2.3.12}$$

Proof Without loss of generality, we may assume that $\big(x_\alpha(t), t\big) = (0,0)$ and that α is isolated, i.e. $v^+ = v_\alpha^+$ and $v^- = v_\alpha^-$. Let α be a shock wave. The parameterization (1.1.15) implies that for $s_{k_\alpha}(v^-, v^+) < \dot{x}_\alpha$,

$$\widetilde{p}_{k_\alpha}(\xi) = \begin{cases} 0, & \text{if } \xi < s_{k_\alpha}(v^-, v^+) \\ \sigma_\alpha, & \text{if } s_{k_\alpha}(v^-, v^+) < \xi < \dot{x}_\alpha \\ 0, & \text{if } \dot{x}_\alpha < \xi. \end{cases}$$

From this, we verify that

$$D_\alpha(t) = \int_{-\infty}^{\infty} |\widetilde{p}_{k_\alpha}(\xi)|\, d\xi = |\sigma_\alpha|\big|\dot{x}_\alpha - s_{k_\alpha}(v^-, v^+)\big|.$$

A similar calculation demonstrates the result when $\dot{x}_\alpha \leq s_{k_\alpha}(v^-, v^+)$.

Suppose now that α is a rarefaction wave. We begin by assuming that $\dot{x}_\alpha \in \big[\lambda_{k_\alpha}(v^-), \lambda_{k_\alpha}(v^+)\big]$. The parameterization of rarefaction curves (1.1.15) implies that, when $\lambda_{k_\alpha}(v^-) \le \xi \le \dot{x}_\alpha$,

$$\widetilde{p}_{k_\alpha}(\xi) = \lambda_{k_\alpha}\big(V_r(\xi)\big) - \lambda_{k_\alpha}(v^-) = \xi - \lambda_{k_\alpha}(v^-),$$

and, when $\dot{x}_\alpha \le \xi \le \lambda_{k_\alpha}(v^+)$,

$$\widetilde{p}_{k_\alpha}(\xi) = \lambda_{k_\alpha}(v^+) - \lambda_{k_\alpha}\big(V_r(\xi)\big) = \lambda_{k_\alpha}(v^+) - \xi.$$

With these identities, we compute

$$\begin{aligned}
D_\alpha(t) &= \int_{-\infty}^{\infty} |\widetilde{p}_{k_\alpha}(\xi)|\, d\xi \\
&= \int_{\lambda_{k_\alpha}(v^-)}^{\dot{x}_\alpha} |\xi - \lambda_{k_\alpha}(v^-)|\, d\xi + \int_{\dot{x}_\alpha}^{\lambda_{k_\alpha}(v^+)} |\lambda_{k_\alpha}(v^+) - \xi|\, d\xi \\
&= \frac{1}{2}(\dot{x}_\alpha - \lambda_{k_\alpha}(v^-))^2 + \frac{1}{2}(\lambda_{k_\alpha}(v^+) - \dot{x}_\alpha)^2.
\end{aligned}$$

Now, suppose that $\dot{x}_\alpha < \lambda_{k_\alpha}(v^-)$.

$$\begin{aligned}
D_\alpha(t) &= \int_{-\infty}^{\infty} |\widetilde{p}_{k_\alpha}(\xi)|\, d\xi \\
&= \int_{\dot{x}_\alpha}^{\lambda_{k_\alpha}(v^-)} |\sigma_\alpha|\, d\xi + \int_{\lambda_{k_\alpha}(v^-)}^{\lambda_{k_\alpha}(v^+)} \big|\lambda_{k_\alpha}(v^+) - \xi\big|\, d\xi \\
&= |\sigma_\alpha|\big|\dot{x}_\alpha - \lambda_{k_\alpha}(v^-)\big| + \frac{1}{2}\big(\lambda_{k_\alpha}(v^-) - \lambda_{k_\alpha}(v^+)\big)^2 \qquad (2.3.13) \\
&= |\sigma_\alpha|\big|\dot{x}_\alpha - \lambda_{k_\alpha}(v^-)\big| + |\sigma_\alpha|\Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(v^+)\big) - \lambda_{k_\alpha}(v^-)\Big) \\
&= |\sigma_\alpha|\Big|\dot{x}_\alpha - \frac{1}{2}\big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(v^+)\big)\Big|
\end{aligned}$$

A similar proof holds when $\lambda_{k_\alpha}(v^+) \le \dot{x}_\alpha$. ■

As we remarked earlier in (2.1.14), the wave speeds satisfy $|\dot{x}_\alpha| \le \Lambda$, $\forall \alpha$. Using our explicit forms of the discrepancy, we can verify that

$$D_\alpha(t) = \mathcal{O}(1)|\sigma_\alpha|, \qquad \forall \alpha \in \mathcal{W}, \tag{2.3.14}$$

where $\mathcal{O}(1)$ is a function with an upper bound in Ω computable entirely from the system (1.1.2). For rarefaction waves, we will need the more precise estimates of the next lemma.

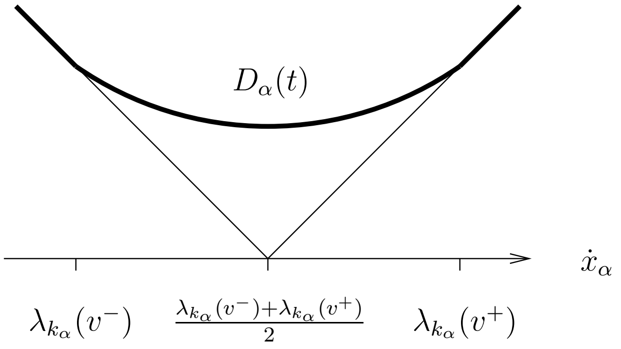
$D_\alpha(t)$
$\dot{x}_\alpha$
$\lambda_{k_\alpha}(v^-)$
$\frac{\lambda_{k_\alpha}(v^-)+\lambda_{k_\alpha}(v^+)}{2}$
$\lambda_{k_\alpha}(v^+)$

Figure 2.3: The bold line is the graph of $D_\alpha(t)$ as a function of $\dot{x}_\alpha$.

Lemma 2.3.2 *Assume that α is a rarefaction wave. For any speed $\dot{x}_\alpha$ we have*

$$|\sigma_\alpha|\Big|\dot{x}_\alpha - \frac{1}{2}\big(\lambda_{k_\alpha}(v_\alpha^-) + \lambda_{k_\alpha}(v_\alpha^+)\big)\Big| \leq D_\alpha(t) \tag{2.3.15}$$

with equality precisely when $\dot{x}_\alpha \notin \big(\lambda_{k_\alpha}(v_\alpha^-), \lambda_{k_\alpha}(v_\alpha^+)\big)$. Moreover,

$$\inf_{\dot{x}_\alpha} D_\alpha(t) = \frac{|\sigma_\alpha|^2}{4}. \tag{2.3.16}$$

Proof We will assume that the wave is isolated and hence, that $v^+ = v_\alpha^+$ and $v^- = v_\alpha^-$. Refer to Figure 2.3 for an illustration of the discrepancy $D_\alpha(t)$. From Lemma 2.3.1, we verify that the left and right hand sides of (2.3.15) coincide when $\dot{x}_\alpha \notin \big(\lambda_{k_\alpha}(v^-), \lambda_{k_\alpha}(v^+)\big)$. For $\dot{x}_\alpha \in \big(\lambda_{k_\alpha}(v^-), \lambda_{k_\alpha}(v^+)\big)$ we evaluate the derivative of both sides of expression (2.3.15):

$$\frac{d}{d\dot{x}_\alpha}\Big(|\sigma_\alpha|\Big|\dot{x}_\alpha - \frac{1}{2}\big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(v^+)\big)\Big|\Big) = \pm|\sigma_\alpha|, \tag{2.3.17}$$

$$\frac{d}{d\dot{x}_\alpha}D_\alpha(t) = 2\Big(\dot{x}_\alpha - \frac{1}{2}\big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(v^+)\big)\Big). \tag{2.3.18}$$

For $\lambda_{k_\alpha}(v^-) < \dot{x}_\alpha < \big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(v^+)\big)/2$, these derivatives evaluate to

$$-|\sigma_\alpha| < \frac{d}{d\dot{x}_\alpha}D_\alpha(t),$$

and the desired inequality (2.3.15) is clearly strict. For $\big(\lambda_{k_\alpha}(v^-)+\lambda_{k_\alpha}(v^+)\big)/2 < \dot{x}_\alpha < \lambda_{k_\alpha}(v^+)$, the derivative of the discrepancy is strictly smaller than the derivative of the left hand side of (2.3.15),

$$\frac{d}{d\dot{x}_\alpha} D_\alpha(t) < |\sigma_\alpha|. \tag{2.3.19}$$

Since both terms agree at $\dot{x}_\alpha = \lambda_{k_\alpha}(v^+)$, inequality (2.3.15) must be strict or else, by the mean value theorem, inequality (2.3.19) would be violated for at least one value of $\dot{x}_\alpha$.

According to formula (2.3.18) the minimum of $D_\alpha(t)$ occurs at

$$\dot{x}_\alpha = \big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(v^+)\big)/2.$$

Explicitly evaluating equation (2.3.12) at this minimum demonstrates (2.3.16). ■

Chapter 3

Estimate for Scalar Conservation Laws

This chapter is dedicated to a proof of an a posteriori error estimate for scalar conservation laws. This estimate is closely related to the estimate of Chapter 4 but is of independent interest for three reasons. First of all, the estimate applies to a class of approximations wider than those of the front-tracking method. Secondly, the estimate coincides with the result of Cockburn and Gau, Theorem 1.2.1. Finally, the proof proceeds in the same manner as the involved proof for systems of conservation laws and therefore provides an introduction to the methods of Chapter 4.

3.1 Discrepancies and Cockburn's Estimators

We now specialize the results of Section 2.3 to scalar conservation laws and demonstrate the equivalence between the phase-space discrepancies D_α^* and Cockburn's error estimators (1.2.4). The demonstration of Lemma 3.1.1 parallels that of Lemmas 2.3.1 and 2.3.2.

Lemma 3.1.1 *Consider a scalar conservation law* (1.1.2) *satisfying* $f'' > 0$*. If* $\alpha \in \mathcal{W}\big(v(t)\big)$ *is a shock wave then*

$$D_\alpha^*(t) = |v^- - v^+|\big|\dot{x}_\alpha - s(v^-, v^+)\big|. \tag{3.1.1}$$

If α *is a rarefaction wave,* $I \equiv \big[f'(v^-), f'(v^+)\big]$*, and* g_α *satisfies* $f'(g_\alpha) = \dot{x}_\alpha$*, then*

$$D_\alpha^*(t) = \begin{cases} \big|g_\alpha - v^-\big|\big[\dot{x}_\alpha - s(g_\alpha, v^-)\big] - \big|v^+ - g_\alpha\big|\big[\dot{x}_\alpha - s(v^+, g_\alpha)\big], & \textit{if } \dot{x}_\alpha \in I \\ |v^- - v^+|\big|\dot{x}_\alpha - s(v^-, v^+)\big|, & \textit{otherwise.} \end{cases} \tag{3.1.2}$$

Moreover, for a rarefaction wave

$$|v^- - v^+|\big|\dot{x}_\alpha - s(v^-, v^+)\big| \le D^*_\alpha(t). \tag{3.1.3}$$

Proof Suppose that α is a shock wave located at $\big(t, x_\alpha(t)\big)$ in the approximation v. In this case,

$$V(\xi) = \begin{cases} v^- & \text{for } \xi \le \dot{x}_\alpha(t), \\ v^+ & \text{for } \dot{x}_\alpha(t) < \xi, \end{cases} \tag{3.1.4}$$

and according to (1.1.11),

$$V_r(\xi) = \begin{cases} v^- & \text{for } \xi \le s(v^-, v^+), \\ v^+ & \text{for } s(v^-, v^+) < \xi. \end{cases} \tag{3.1.5}$$

With these two formulas we compute

$$D^*_\alpha(t) = \int_{-\infty}^{\infty} \big|V(\xi) - V_r(\xi)\big| \, d\xi = |v^- - v^+|\big|\dot{x}_\alpha - s(v^-, v^+)\big|.$$

Suppose now that α is a rarefaction wave. The function $V(\xi)$ remains the same as in (3.1.4). The Riemann solution V_r is the continuous solution described in the second part of Theorem 1.1.1. Since $f'' > 0$, f' is monotone and $g \equiv (f')^{-1}$ is well-defined. The continuous rarefaction solution can therefore be written as

$$V_r(\xi) = \begin{cases} v^- & \text{if } \xi < f'(v^-), \\ g(\xi) & \text{if } \xi \in I, \\ v^+ & \text{if } f'(v^+) < \xi. \end{cases} \tag{3.1.6}$$

To compute $D^*_\alpha(t)$, we need to consider three cases depending on $\dot{x}_\alpha$. Begin by assuming that $\dot{x}_\alpha < f'(v^-)$. In this case,

$$\begin{aligned} D^*_\alpha(t) &= \int_{-\infty}^{\infty} \big|V(\xi) - V_r(\xi)\big| \, d\xi \\ &= \int_{\dot{x}_\alpha}^{f'(v^-)} |v^+ - v^-| \, d\xi + \int_{f'(v^-)}^{f'(v^+)} \big|v^+ - g(\xi)\big| \, d\xi. \end{aligned}$$

For the second integral, use the change of variable $\xi = f'(v)$ and integration by parts to compute

$$\begin{aligned} D^*_\alpha(t) =&|v^+ - v^-|\big|f'(v^-) - \dot{x}_\alpha\big| + \int_{v^-}^{v^+} (v^+ - v) f''(v)\ dv \\ =&|v^+ - v^-|\big|f'(v^-) - \dot{x}_\alpha\big| + (v^+ - v)f'(v)\Big|_{v^-}^{v^+} + \int_{v^-}^{v^+} f'(v)\ dv \\ =&|v^+ - v^-|\big|f'(v^-) - \dot{x}_\alpha\big| - (v^+ - v^-)f'(v^-) + \big(f(v^+) - f(v^-)\big) \\ =&|v^+ - v^-|\big|f'(v^-) - \dot{x}_\alpha\big| + |v^+ - v^-|\big|s(v^+, v^-) - f'(v^-)\big| \\ =&|v^+ - v^-|\big|\dot{x}_\alpha - s(v^-, v^+)\big|. \end{aligned}$$

A similar calculation demonstrates the result for $f'(v^+) < \dot{x}_\alpha$.

We now consider the case where $\dot{x}_\alpha \in I$. We begin the computation of D^*_α by performing the change of variable $\xi = f'(v)$.

$$\begin{aligned} D^*_\alpha(t) &= \int_{-\infty}^{\infty} \big|V(\xi) - V_r(\xi)\big|\ d\xi \\ &= \int_{f'(v^-)}^{\dot{x}_\alpha} \big|g(\xi) - v^-\big|\ d\xi + \int_{\dot{x}_\alpha}^{f'(v^+)} \big|v^+ - g(\xi)\big|\ d\xi \\ &= \int_{v^-}^{g(\dot{x}_\alpha)} (v - v^-) f''(v)\ dv + \int_{g(\dot{x}_\alpha)}^{v^+} (v^+ - v) f''(v)\ dv. \end{aligned}$$

Using integration by parts on each term and noticing that $f'\big(g(\dot{x}_\alpha)\big) = \dot{x}_\alpha$, we find

$$\begin{aligned} D^*_\alpha(t) =&(v - v^-)f'(v)\Big|_{v^-}^{g(\dot{x}_\alpha)} - \int_{v^-}^{g(\dot{x}_\alpha)} f'(v)\ dv \\ &+ (v^+ - v)f'(v)\Big|_{g(\dot{x}_\alpha)}^{v^+} + \int_{g(\dot{x}_\alpha)}^{v^+} f'(v)\ dv \\ =&\big(g(\dot{x}_\alpha) - v^-\big)\dot{x}_\alpha - \Big(f\big(g(\dot{x}_\alpha)\big) - f(v^-)\Big) \\ &- \big(v^+ - g(\dot{x}_\alpha)\big)\dot{x}_\alpha + \Big(f(v^+) - f\big(g(\dot{x}_\alpha)\big)\Big) \\ =&\big|g(\dot{x}_\alpha) - v^-\big|\Big[\dot{x}_\alpha - s\big(g(\dot{x}_\alpha), v^-\big)\Big] - \big|v^+ - g(\dot{x}_\alpha)\big|\Big[\dot{x}_\alpha - s\big(v^+, g(\dot{x}_\alpha)\big)\Big]. \end{aligned}$$

This demonstrates formula (3.1.2) with $g_\alpha \equiv g(\dot{x}_\alpha)$.

We now demonstrate inequality (3.1.3). According to (3.1.2), it suffices to prove that for $\dot{x}_\alpha \in I$, the expression

$$|v^- - v^+|\big|\dot{x}_\alpha - s(v^-, v^+)\big| \tag{3.1.7}$$

is smaller than

$$\big|g_\alpha - v^-\big|\big[\dot{x}_\alpha - s(g_\alpha, v^-)\big] - \big|v^+ - g_\alpha\big|\big[\dot{x}_\alpha - s(v^+, g_\alpha)\big]. \tag{3.1.8}$$

The derivative with respect to $\dot{x}_\alpha$ of expression (3.1.7) is

$$\frac{d}{d\dot{x}_\alpha}\Big[|v^- - v^+|\big|\dot{x}_\alpha - s(v^-, v^+)\big|\Big] = \begin{cases} -|v^+ - v^-| & \text{for } \dot{x}_\alpha \leq s(v^-, v^+), \\ |v^+ - v^-| & \text{for } s(v^-, v^+) < \dot{x}_\alpha. \end{cases} \tag{3.1.9}$$

Since $g = (f')^{-1}$, the derivative of (3.1.8) is

$$\begin{aligned} \frac{d}{d\dot{x}_\alpha} D_\alpha(t) &= \frac{d}{d\dot{x}_\alpha}\Big[\big|g_\alpha - v^-\big|\big(\dot{x}_\alpha - s(g_\alpha, v^-)\big) - \big|v^+ - g_\alpha\big|\big(\dot{x}_\alpha - s(v^+, g_\alpha)\big)\Big] \\ &= \frac{d}{d\dot{x}_\alpha}\Big[\Big(f(v^+) + f(v^-) - 2f\big(g(\dot{x}_\alpha)\big)\Big) - \dot{x}_\alpha\big(v^+ + v^- - 2g(\dot{x}_\alpha)\big)\Big] \\ &= -2f'\big(g(\dot{x}_\alpha)\big)g'(\dot{x}_\alpha) - \big(v^+ + v^- - 2g(\dot{x}_\alpha)\big) + 2\dot{x}_\alpha g'(\dot{x}_\alpha) \\ &= 2g(\dot{x}_\alpha) - v^+ - v^- = 2\left(g(\dot{x}_\alpha) - \frac{v^+ + v^-}{2}\right). \end{aligned} \tag{3.1.10}$$

The functions (3.1.7) and (3.1.8) attain their minimas at different values of $\dot{x}_\alpha$, namely $s(v^-, v^+)$ and $f'\big((v^+ + v^-)/2\big)$.

For definiteness, suppose that $s(v^-, v^+) < f'\big((v^+ + v^-)/2\big)$. When the velocity $\dot{x}_\alpha = f'(v^-)$, then both (3.1.7) and (3.1.8) are equal. When $\dot{x}_\alpha \in \big[f'(v^-), s(v^-, v^+)\big]$, the derivative (3.1.9) is smaller than (3.1.10) and therefore (3.1.7) must be smaller than (3.1.8). Similarly, expressions (3.1.7) and (3.1.8) agree at $\dot{x}_\alpha = f'(v^+)$ and the derivative of (3.1.7) is greater than the derivative of (3.1.8). This implies that expression (3.1.7) must be smaller than (3.1.8) for all $\dot{x}_\alpha \in [f'\big((v^+ + v^-)/2\big), f'(v^+)\big]$. Inside the interval $\big[s(v^-, v^+) < f'\big((v^+ + v^-)/2\big)\big]$ (3.1.7) is increasing but (3.1.8) is decreasing. Having just demonstrated the inequality at $\dot{x}_\alpha = f'\big((v^+ + v^-)/2\big)$, the inequality must hold over the middle interval. ■

In Cockburn and Gau's a posteriori error estimate, Theorem 1.2.1, the numerical error is measured by the residual in smooth regions and by quantities $\{R_i\}$ defined along the hyperplanes $\{H_i\}$ of discontinuity. The latter are integrated against the volume form of the hyperplane H_i. On the other hand, in our estimate the phase-space discrepancies D^*_α of a discontinuity appear integrated over time. The following theorem shows that these notions measure the same local error.

Theorem 3.1.1 *Let v be a piecewise smooth function and let* (1.1.2) *be a scalar conservation law satisfying $f'' > 0$. Suppose that α is a wave in v whose trajectory $\big(t, x_\alpha(t)\big)$ for $t \in [t_1, t_2]$ traces the hyperplane $H_\alpha \subset \mathbb{R}^+ \times \mathbb{R}$. Then*

$$\big|\big|R_\alpha(v)\big|\big|_{L^1(H_\alpha)} = \int_{t_1}^{t_2} D^*_\alpha(t)\ dt. \tag{3.1.11}$$

Proof Parameterize by arclength s the trajectory of the wave α, $\gamma(s) = \big(t(s), x_\alpha(s)\big)$. If the velocity $\dot{x}_\alpha$ of the wave is positive then the unit normal with a negative time component is

$$\vec{n}_\alpha = \left(\frac{1}{\sqrt{1+(\dot{x}_\alpha)^2}}, \frac{-\dot{x}_\alpha}{\sqrt{1+(\dot{x}_\alpha)^2}}\right). \tag{3.1.12}$$

Recall that in the definition of the error estimator $R_\alpha(v)$, for any function G the jump across a discontinuity is

$$\big[G(v)\big]_{H_i}(x) = \lim_{\epsilon \longrightarrow 0} G\big(v(x+\epsilon\vec{n}_{H_i})\big) - G\big(v(x-\epsilon\vec{n}_{H_i})\big). \tag{3.1.13}$$

For $s \in [s_1, s_2]$ and any c, the integral of the non-trivial component of (1.2.4) is

$$\begin{aligned}\int_{H_\alpha} R_\alpha(v)(\gamma(s))\ ds &= \int_{s_1}^{s_2} \Big(\big[F(v,c)\big]_{H_\alpha}, \big[|u-c|\big]_{H_\alpha}\Big)\cdot\vec{n}_\alpha\ ds \\ &= \int_{s_1}^{s_2}\left\{\big[F(v,c)\big]_{H_\alpha}\frac{1}{\sqrt{1+(\dot{x}_\alpha)^2}} - \big[|u-c|\big]_{H_\alpha}\frac{\dot{x}_\alpha}{\sqrt{1+(\dot{x}_\alpha)^2}}\right\} ds\end{aligned} \tag{3.1.14}$$

Since

$$\gamma^*(dt) = \frac{dt}{ds}\ ds = \frac{1}{\sqrt{1+(\dot{x}_\alpha)^2}}\ ds, \text{ and } \gamma^*(dx) = \frac{dx}{ds}\ ds = \frac{\dot{x}_\alpha}{\sqrt{1+(\dot{x}_\alpha)^2}}\ ds,$$

the integral in s can be written as a line integral over the path γ and then as a time integral.

$$\begin{aligned}\int_{s_1}^{s_2} \left\{ \big[F(v,c)\big]_{H_\alpha} \frac{1}{\sqrt{1+(\dot{x}_\alpha)^2}} - \big[|u-c|\big]_{H_\alpha} \frac{\dot{x}_\alpha}{\sqrt{1+(\dot{x}_\alpha)^2}} \right\} ds \\ = \int_\gamma \big[F(v,c)\big]_{H_\alpha}\, dt - \big[|u-c|\big]_{H_\alpha}\, dx \\ = \int_{t_1}^{t_2} \left\{ \big[F(v,c)\big]_{H_\alpha} - \dot{x}_\alpha \big[|u-c|\big]_{H_\alpha} \right\} dt \qquad (3.1.15)\end{aligned}$$

The integrand of (3.1.15) is the error estimator of Cockburn as a function of time. To demonstrate (3.1.11), it therefore suffices to show that the integrand in (3.1.15) is equal pointwise to $D^*_\alpha(t)$.

We remark that the same calculation could have been performed with $\dot{x}_\alpha$ negative. In this case, we would have obtained the opposite of (3.1.12). On the other hand, by reversing the direction of $\vec{n}_\alpha$ we would also have reversed the order in which we measure the jump across the discontinuity, defined in (3.1.13). This jump would have changed by a negative sign hence resulting in the same formula (3.1.15). For this reason, in equation (3.1.15) it suffices to use the definition (3.1.13) for $\dot{x}_\alpha$ positive.

Suppose that α is a shock wave, i.e. a discontinuity in v satisfying $v^+ < v^-$. As a function of t, the integrand $R_\alpha(v)$ is

$$\begin{aligned}R_\alpha(v)(t) &= \max\left\{0, \sup_c \left\{ \big[F(v,c)\big]_{H_\alpha} - \dot{x}_\alpha \big[|v-c|\big]_{H_\alpha} \right\}\right\} \\ &= \max\left\{0, \sup_{v^+<v^-\le c} \left\{ |v^+-c|\big(s(v^+,c)-\dot{x}_\alpha\big) - |v^--c|\big(s(v^-,c)-\dot{x}_\alpha\big) \right\}, \right. \\ &\qquad \sup_{c\le v^+<v^-} \left\{ |v^+-c|\big(s(v^+,c)-\dot{x}_\alpha\big) - |v^--c|\big(s(v^-,c)-\dot{x}_\alpha\big) \right\}, \\ &\qquad \left. \sup_{v^+<c<v^-} \left\{ |v^+-c|\big(s(v^+,c)-\dot{x}_\alpha\big) - |v^--c|\big(s(v^-,c)-\dot{x}_\alpha\big) \right\} \right\} \\ &= \max\left\{0, \big(f(v^-)-f(v^+)\big) - \dot{x}_\alpha(v^- - v^+), \right. \\ &\qquad \big(f(v^+)-f(v^-)\big) - \dot{x}_\alpha(v^+ - v^-), \\ &\qquad \left. \sup_{v^+<c<v^-} \left\{ \big(2f(c)-f(v^+)-f(v^-)\big) - \dot{x}_\alpha\big(2c - v^+ - v^-\big) \right\} \right\}\end{aligned}$$

(3.1.16)

The maximum of the first three terms in (3.1.16) is clearly

$$|v^- - v^+|\big|s(v^-,v^+) - \dot{x}_\alpha\big|. \tag{3.1.17}$$

To find the supremum over $c \in [v^+, v^-]$ of the last expression in (3.1.16), we differentiate it in c to find

$$\frac{d}{dc}\Big(\big(2f(c) - f(v^+) - f(v^-)\big) - \dot{x}_\alpha\big(2c - v^+ - v^-\big)\Big) = 2\big(f'(c) - \dot{x}_\alpha\big). \tag{3.1.18}$$

Depending on $\dot{x}_\alpha$, the calculation of this supremum is subdivided into three cases. In the first case, $f'(v^-) \leq \dot{x}_\alpha$, the derivative (3.1.18) is always negative and the supremum occurs when $c = v^+$. Since $s(v^-, v^+) < f'(v^-) \leq \dot{x}_\alpha$, this supremum is

$$\big(f(v^+) - f(v^-)\big) - \dot{x}_\alpha(v^+ - v^-) = |v^- - v^+|\big|s(v^-,v^+) - \dot{x}_\alpha\big|. \tag{3.1.19}$$

In the second case, we assume that $\dot{x}_\alpha \leq f'(v^+)$. For these values of $\dot{x}_\alpha$, the derivative (3.1.19) is always positive and the supremum occurs when $c = v^-$. Evaluating this supremum, we obtain

$$\big(f(v^-) - f(v^+)\big) - \dot{x}_\alpha(v^- - v^+) = |v^- - v^+|\big|s(v^-,v^+) - \dot{x}_\alpha\big|. \tag{3.1.20}$$

Finally, when $\dot{x}_\alpha \in I$, the derivative (3.1.18) over the interval $[v^+, v^-]$ is first negative and then positive over I. The maximum over c must therefore be located at the boundaries of $[v^+, v^-]$. Evaluating the fourth term of (3.1.16) at both boundaries we find

$$\begin{aligned}
\sup_{v^+<c<v^-} & \Big\{\big(2f(c) - f(v^+) - f(v^-)\big) - \dot{x}_\alpha\big(2c - v^+ - v^-\big)\Big\} \\
&= \max\Big\{\big(f(v^+) - f(v^-)\big) - \dot{x}_\alpha(v^+ - v^-), \big(f(v^-) - f(v^+)\big) - \dot{x}_\alpha(v^- - v^+)\Big\} \\
&= |v^- - v^+|\big|s(v^-,v^+) - \dot{x}_\alpha\big|. \qquad (3.1.21)
\end{aligned}$$

From expressions (3.1.17), (3.1.19), (3.1.20), and (3.1.21) we conclude that, no matter the value of c or $\dot{x}_\alpha$,

$$R_\alpha(v)(t) = |v^- - v^+|\big|s(v^-,v^+) - \dot{x}_\alpha\big|. \tag{3.1.22}$$

This is precisely the phase-space discrepancy of a shock wave $D^*_\alpha(t)$, as described in Lemma (3.1.1).

Suppose now that α is a rarefaction wave. The neighboring states therefore satisfy $v^- < v^+$.

$$\begin{aligned}
R_\alpha(v)(t) &= \max\Big\{0, \sup_c \Big\{ \big[F(v,c)\big]_{H_\alpha} - \dot{x}_\alpha \big[|v-c|\big]_{H_\alpha} \Big\}\Big\} \\
&= \max\Big\{0, \sup_{v^+<v^-\le c} \Big\{ |v^+-c|\big(s(v^+,c)-\dot{x}_\alpha\big) - |v^- - c|\big(s(v^-,c)-\dot{x}_\alpha\big)\Big\}, \\
&\qquad \sup_{c\le v^+<v^-} \Big\{ |v^+-c|\big(s(v^+,c)-\dot{x}_\alpha\big) - |v^- - c|\big(s(v^-,c)-\dot{x}_\alpha\big)\Big\}, \\
&\qquad \sup_{v^+<c<v^-} \Big\{ |v^+-c|\big(s(v^+,c)-\dot{x}_\alpha\big) - |v^- - c|\big(s(v^-,c)-\dot{x}_\alpha\big)\Big\}\Big\} \\
&= \max\Big\{0, \big(f(v^-)-f(v^+)\big) - \dot{x}_\alpha(v^- - v^+), \\
&\qquad \big(f(v^+)-f(v^-)\big) - \dot{x}_\alpha(v^+ - v^-), \\
&\qquad \sup_{v^+<c<v^-} \Big\{ \big(f(v^+)+f(v^-)-2f(c)\big) - \dot{x}_\alpha\big(v^+ + v^- - 2c\big)\Big\}\Big\}
\end{aligned} \tag{3.1.23}$$

The maximum of the first three expressions is

$$|v^- - v^+|\big|s(v^-,v^+) - \dot{x}_\alpha\big|. \tag{3.1.24}$$

Compute the derivative with respect to c of the last expression in (3.1.23). This gives

$$\frac{d}{dc}\Big(\big(f(v^+)+f(v^-)-2f(c)\big) - \dot{x}_\alpha\big(v^+ + v^- - 2c\big)\Big) = 2\big(\dot{x}_\alpha - f'(c)\big). \tag{3.1.25}$$

Again we consider three cases depending on the velocity $\dot{x}_\alpha$. When $f'(v^+) \le \dot{x}_\alpha$, the supremum occurs at $c = v^+$ and when $\dot{x}_\alpha \le f'(v^-)$, the supremum occurs at $c = v^-$. Therefore in both these cases the supremum is

$$|v^- - v^+|\big|s(v^-,v^+) - \dot{x}_\alpha\big|. \tag{3.1.26}$$

Suppose now that $\dot{x}_\alpha \in I$. Formula (3.1.25) shows that the supremum is located at the value $c = g_\alpha$ for which $f'(g_\alpha) = \dot{x}_\alpha$. Evaluating the last term in (3.1.23) at its supremum $c = g_\alpha$, we find

$$|v^+ - g_\alpha|\big(s(v^+, g_\alpha) - \dot{x}_\alpha\big) - |v^- - g_\alpha|\big(s(v^-, g_\alpha) - \dot{x}_\alpha\big), \tag{3.1.27}$$

which is precisely the discrepancy $D^*_\alpha(t)$ given in (3.1.2). For $\dot{x}_\alpha \in I$, the error estimator $R_\alpha(v)$ is the maximum of (3.1.24) and (3.1.27) but from inequality (3.1.3), we know that (3.1.24) is less than (3.1.27). From these observations we conclude that

$$R_\alpha(v)(t) = \begin{cases} \big|g_\alpha - v^-\big|\big(\dot{x}_\alpha - s(g_\alpha, v^-)\big) - \big|v^+ - g_\alpha\big|\big(\dot{x}_\alpha - s(v^+, g_\alpha)\big), & \text{if } \dot{x}_\alpha \in I \\ |v^- - v^+|\big|\dot{x}_\alpha - s(v^-, v^+)\big|, & \text{otherwise.} \end{cases} \tag{3.1.28}$$

Comparing with (3.1.2) demonstrates the result. ■

3.2 Main Result

In this section we demonstrate an a posteriori error estimate for piecewise smooth approximate solutions to nonlinear scalar conservation laws. The estimate coincides with Cockburn and Gau's Theorem 1.2.1 although their estimate also applied to approximations in several space dimensions. Despite their equivalence, the estimate of Cockburn and Gau is constructed from the theory of Kuznetsov while our result is based on the L^1 stability theory of Keyfitz [28].

Theorem 3.2.1 *Consider a scalar conservation law* (1.1.2) *satisfying* $f'' > 0$*. Let* u *be a piecewise smooth entropy-admissible solution to* (1.1.2) *whose discontinuities satisfy the Rankine-Hugoniot condition. Let* v *be a piecewise smooth function satisfying*

$$|\dot{x}_\alpha| \leq \sup |f'|, \qquad \forall \alpha \in \mathcal{W}(v(t)),$$

and where the supremum occurs over the convex hull of the range of $u(0,\cdot)$*. If both* u *and* v *have a finite number of discontinuities and*

$$\big|\big|u(t,\cdot) - v(t,\cdot)\big|\big|_{L^1} < \infty,\ \forall t,$$

then

$$\begin{aligned} \big|\big|u(t,\cdot) - v(t,\cdot)\big|\big|_{L^1} \ \leq\ & \big|\big|u(0,\cdot) - v(0,\cdot)\big|\big|_{L^1} + \big|\big|v_t + f(v)_x\big|\big|_{L^1(\mathbb{R}\times[0,t])} \\ & + \int_0^t \sum_{\alpha \in \mathcal{W}(v(s))} D^*_\alpha(s)\ ds. \end{aligned} \tag{3.2.1}$$

Proof We begin by assuming that the discontinuous fronts in both solutions are disjoint at time t. Partition space into intervals $\big(z_i(t), z_{i+1}(t)\big)$ over which $u - v$ is smooth and has but one sign. Let $\big(z_0(t), z_1(t)\big) = \big(-\infty, z_1(t)\big)$ and $\big(z_m(t), z_{m+1}(t)\big) = \big(z_m(t), \infty\big)$ be the intervals extending to positive and negative infinity. At a point $z_i(t)$, either u crosses v smoothly or one of the functions has a discontinuity. For convenience, we will write

$$\lim_{x \to z_i \pm} u(t,x) = u_i^{\pm} \text{ and } \lim_{x \to z_i \pm} v(t,x) = v_i^{\pm},$$

and in general suppress the time dependence. From the scalar conservation law (1.1.2) and the definition of the residual we have

$$u_t = -f(u)_x, \qquad \text{and} \qquad v_t = -f(v)_x + R(v). \tag{3.2.2}$$

We write the time derivative of the L^1-norm as a sum over the partition of space

$$\begin{aligned}
\frac{d}{dt}\big|\big|u(t,\cdot) - v(t,\cdot)\big|\big|_{L^1} =& \sum_{i=0}^{m} \operatorname{sign}(u-v) \frac{d}{dt} \int_{z_i(t)}^{z_{i+1}(t)} \big(u(t,x) - v(t,x)\big)\, dx \\
=& \sum_{i=1}^{m-1} \bigg\{ \operatorname{sign}(u-v) \int_{z_i}^{z_{i+1}} \big(u_t - v_t\big)\, dx \\
& \qquad\qquad + \big|u_{i+1}^- - v_{i+1}^-\big| \dot{z}_{i+1} - \big|u_i^- - v_i^-\big| \dot{z}_i \bigg\} \\
& + \operatorname{sign}(u_1^- - v_1^-) \int_{-\infty}^{z_1} \big(u_t - v_t\big)\, dx + \big|u_1^- - v_1^-\big| \dot{z}_1 \\
& + \operatorname{sign}(u_m^+ - v_m^+) \int_{z_m}^{\infty} \big(u_t - v_t\big)\, dx + \big|u_m^+ - v_m^+\big| \dot{z}_m.
\end{aligned} \tag{3.2.3}$$

First use the identities (3.2.2) to replace the time derivatives with space derivatives and then apply the fundamental theorem of calculus to rewrite (3.2.3) as

$$\begin{aligned}
& \sum_{i=1}^{m-1} \bigg\{ \operatorname{sign}(u-v) \int_{z_i}^{z_{i+1}} \big(-f(u)_x + f(v)_x - R(v)\big)\, dx \\
& \qquad + \big|u_{i+1}^- - v_{i+1}^-\big| \dot{z}_{i+1} - \big|u_i^- - v_i^-\big| \dot{z}_i \bigg\} \\
& + \operatorname{sign}(u_1^- - v_1^-) \int_{-\infty}^{z_1} \big(-f(u)_x + f(v)_x - R(v)\big)\, dx + \big|u_1^- - v_1^-\big| \dot{z}_1 \\
& + \operatorname{sign}(u_m^+ - v_m^+) \int_{z_m}^{\infty} \big(-f(u)_x + f(v)_x - R(v)\big)\, dx + \big|u_m^+ - v_m^+\big| \dot{z}_m
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m-1} \Big\{ \operatorname{sign}(u-v)\Big[-f\big(u_{i+1}^-\big) + f\big(u_i^+\big) + f\big(v_{i+1}^-\big) - f\big(v_i^+\big)\Big] \\
&\qquad\qquad + \big|u_{i+1}^- - v_{i+1}^-\big|\dot z_{i+1} - \big|u_i^- - v_i^-\big|\dot z_i \Big\} - \sum_{i=0}^{m} \operatorname{sign}(u-v) \int_{z_i}^{z_{i+1}} R(v)\ dx \\
&\quad + \operatorname{sign}(u_1^- - v_1^-)\Big[-f\big(u_1^-\big) + f\big(v_1^-\big) + \lim_{x\to-\infty} \Big(f\big(u(t,x)\big) - f\big(v(t,x)\big)\Big)\Big] \\
&\quad + \operatorname{sign}(u_m^+ - v_m^+)\Big[f\big(u_m^+\big) - f\big(v_m^+\big) + \lim_{x\to\infty} \Big(-f\big(u(t,x)\big) + f\big(v(t,x)\big)\Big)\Big] \\
&\quad + \big|u_1^- - v_1^-\big|\dot z_1 + \big|u_m^+ - v_m^+\big|\dot z_m. \qquad (3.2.4)
\end{aligned}
$$

The difference of the piecewise smooth functions u and v is assumed to be in L^1 for all t and therefore their values must agree at infinity. With this fact and formula (1.1.11) for the Rankine-Hugoniot shock speed, we recognize the following upper bound for (3.2.4):

$$
\begin{aligned}
&\frac{d}{dt}\big|\big|u(t,\cdot) - v(t,\cdot)\big|\big|_{L^1} \\
&\quad \le \sum_{i=1}^{m-1} \Big\{ \big|u_{i+1}^- - v_{i+1}^-\big|\big[\dot z_{i+1} - s(u_{i+1}^-, v_{i+1}^-)\big] - \big|u_i^+ - v_i^+\big|\big[\dot z_i - s(u_i^+, v_i^+)\big]\Big\} \\
&\quad + \big|u_1^- - v_1^-\big|\big[\dot z_1 - s(u_1^-, v_1^-)\big] - \big|u_m^+ - v_m^+\big|\big[\dot z_m - s(u_m^+, v_m^+)\big] + \int_{-\infty}^{\infty} \big|R(v)\big|\ dx \\
&= \sum_{i=1}^{m} \Big\{ \big|u_i^- - v_i^-\big|\big[\dot z_i - s(u_i^-, v_i^-)\big] - \big|u_i^+ - v_i^+\big|\big[\dot z_i - s(u_i^+, v_i^+)\big]\Big\} \\
&\quad + \int_{-\infty}^{\infty} \big|R(v)\big|\ dx. \qquad (3.2.5)
\end{aligned}
$$

The next step is to demonstrate that the expression,

$$
\big|u_i^- - v_i^-\big|\big[\dot z_i - s(u_i^-, v_i^-)\big] - \big|u_i^+ - v_i^+\big|\big[\dot z_i - s(u_i^+, v_i^+)\big], \qquad (3.2.6)
$$

is bounded by the discrepancy D_α^* when $z_i = x_\alpha$ for $\alpha \in \mathcal{W}\big(v(t)\big)$ or non-positive otherwise. The proof of this inequality is subdivided into the following cases: either both u and v are smooth at z_i; z_i contains a shock discontinuity in u; or z_i contains a shock or rarefaction wave in v. Moreover, each of these cases is further subdivided into cases depending on changes in $\operatorname{sign}(u-v)$ across z_i. Since the remaining argument is localized at z_i, we will suppress the subscript i.

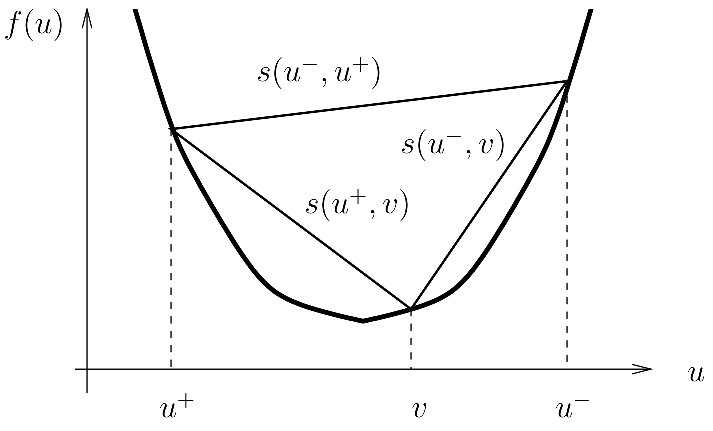

Figure 3.1: The figure indicates that $s(u^+,v) < s(u^-,u^+) < s(u^-,v)$.

In a first instance, suppose that both u and v are continuous at z_i. Then $u-v$ must change sign and the quantities satisfy $u_i^- = v_i^- = u_i^+ = v_i^+$. The term (3.2.6) is zero.

Suppose now that the solution u has a shock discontinuity at z. This shock discontinuity must travel at the Rankine-Hugoniot shock speed $\dot{z} = s(u^-,u^+)$. Assuming that the sign does not change at z, then we verify that

$$\begin{aligned}|u^- - v|\big[\dot{z} - s(u^-,v)\big] &- |u^+ - v|\big[\dot{z} - s(u^+,v)\big]\\ &= \operatorname{sign}(u-v)\Big[(u^- - v)\dot{z} - \big(f(u^-) - f(v)\big)\\ &\qquad - (u^+ - v)\dot{z} + \big(f(u^+) - f(v)\big)\Big]\\ &= |u^- - u^+|\big[\dot{z} - s(u^-,u^+)\big] = 0. \end{aligned} \tag{3.2.7}$$

If the sign of $u-v$ changes at the location of the shock in u, then $u^+ < v < u^-$. Using $\dot{z} = s(u^-,u^+)$, the term (3.2.6) becomes

$$\big|u^- - v\big|\big[s(u^-,u^+) - s(u^-,v)\big] - \big|u^+ - v\big|\big[s(u^-,u^+) - s(u^+,v)\big]. \tag{3.2.8}$$

From Figure 3.1, we see that the convexity of f implies that both terms in (3.2.8) are negative.

We now turn to the case where z contains discontinuities in the approximation v. Assume that $z = x_\alpha$ for a shock wave α. Recall from Lemma 3.1.1 that the phase-space discrepancy of a shock wave is

$$D^*_\alpha = |v^+ - v^-|\big|\dot{x}_\alpha - s(v^+, v^-)\big|. \tag{3.2.9}$$

When the sign of $u - v$ remains unchanged across z, then

$$\begin{aligned}
|u - v^-|\big[\dot{x}_\alpha - s(u, v^-)\big] - {}&|u - v^+|\big[\dot{x}_\alpha - s(u, v^+)\big] \\
&= \operatorname{sign}(u - v)\Big[(u - v^-)\dot{x}_\alpha - \big(f(u) - f(v^-)\big) \\
&\qquad - (u - v^+)\dot{x}_\alpha + \big(f(u) - f(v^+)\big)\Big] \\
&= \operatorname{sign}(u - v)\Big[(v^+ - v^-)\dot{x}_\alpha - \big(f(v^+) - f(v^-)\big)\Big] \\
&\leq |v^+ - v^-|\big|\dot{x}_\alpha - s(v^+, v^-)\big| = D^*_\alpha.
\end{aligned} \tag{3.2.10}$$

If the sign of $u - v$ changes, then

$$\begin{aligned}
&|u - v^-|\big[\dot{x}_\alpha - s(u, v^-)\big] - |u - v^+|\big[\dot{x}_\alpha - s(u, v^+)\big] \\
&\quad = |u - v^-|\big[s(v^-, v^+) - s(u, v^-)\big] - |u - v^+|\big[s(v^-, v^+) - s(u, v^+)\big] \\
&\qquad + \big\{|u - v^-| - |u - v^+|\big\}\big[\dot{x}_\alpha - s(v^-, v^+)\big]
\end{aligned} \tag{3.2.11}$$

The first and second terms of (3.2.10) are negative since the convexity implies the inequalities

$$s(u, v^+) < s(v^-, v^+) < s(u, v^-).$$

By the triangle inequality, the coefficient of the third term in (3.2.11) is bounded by $|v^- - v^+|$. Together with Lemma 3.1.1 these observations imply

$$\begin{aligned}
&|u - v^-|\big[\dot{x}_\alpha - s(u, v^-)\big] - |u - v^+|\big[\dot{x}_\alpha - s(u, v^+)\big] \\
&\qquad\qquad \leq |v^- - v^+|\big|\dot{x}_\alpha - s(v^-, v^+)\big| = D^*_\alpha.
\end{aligned} \tag{3.2.12}$$

Suppose now that the discontinuity at z is a rarefaction wave $\alpha \in \mathcal{W}\big(v(t)\big)$. When the sign of $u - v$ is constant across z, then the same calculation as in (3.2.10) shows that

$$|u-v^-|\big[\dot{x}_\alpha - s(u,v^-)\big] - |u-v^+|\big[\dot{x}_\alpha - s(u,v^+)\big] \le |v^+-v^-|\big|\dot{x}_\alpha - s(v^+,v^-)\big|.$$

According to inequality (3.1.3), this quantity is bounded above by D^*_α. If the sign of $u-v$ changes, then $v^- < u < v^+$ and we rewrite the term (3.2.6) as

$$\begin{aligned} &|u-v^-|\big[\dot{x}_\alpha - s(u,v^-)\big] - |u-v^+|\big[\dot{x}_\alpha - s(u,v^+)\big] \\ &\qquad =\Big[(u-v^-)\dot{x}_\alpha - \big(f(u)-f(v^-)\big)\Big] - \Big[(v^+-u)\dot{x}_\alpha - \big(f(v^+)-f(u)\big)\Big] \\ &\qquad =\big(f(v^+)+f(v^-)-2f(u)\big) - \dot{x}_\alpha\big(v^+ + v^- - 2u\big). \end{aligned} \tag{3.2.13}$$

Since u can be any number in $[v^-,v^+]$, we differentiate in u and search for a critical point.

$$\frac{d}{du}\Big[\big(f(v^+)+f(v^-)-2f(u)\big) - \dot{x}_\alpha\big(v^+ + v^- - 2u\big)\Big] = 2\big(\dot{x}_\alpha - f'(u)\big). \tag{3.2.14}$$

If we introduce the function $g = (f')^{-1}$ then the maximum of (3.2.13) occurs at $g_\alpha \equiv g(\dot{x}_\alpha)$. Substituting $u = g_\alpha$ into (3.2.13) and simplifying, we obtain precisely the equation for the discrepancy described in (3.1.2). Summarizing, we have shown that

$$\begin{aligned} &|u-v^-|\big[\dot{x}_\alpha - s(u,v^-)\big] - |u-v^+|\big[\dot{x}_\alpha - s(u,v^+)\big] \\ &\qquad \le \big|g_\alpha - v^-\big|\big[\dot{x}_\alpha - s(g_\alpha,v^-)\big] - \big|v^+ - g_\alpha\big|\big[\dot{x}_\alpha - s(v^+,g_\alpha)\big] = D^*_\alpha. \end{aligned}$$

Use the estimates for expression (3.2.6) in equation (3.2.5) to obtain the global estimate

$$\frac{d}{dt}\big|\big|u(t,\cdot)-v(t,\cdot)\big|\big|_{L^1} \le \int_{-\infty}^{\infty} \big|R(v)\big|\,dx + \sum_{\alpha\in\mathcal{W}(v(t))} D^*_\alpha(t). \tag{3.2.15}$$

If the fronts of discontinuity intersect transversally at time t, then, by continuity, inequality (3.2.15) still holds. On the other hand, if the fronts coincide then one can perform a small perturbation of the approximations u and v to enforce transversality. Since inequality (3.2.15) depends smoothly on the position and neighboring states of the front, the inequality is preserved as the perturbation converges to the original pair u and v. The inequality therefore holds for all t and we can integrate over $[0, t]$ to complete demonstration of (3.2.1). ■

Chapter 4

Estimate for Systems of Conservation Laws

In this chapter, we demonstrate an a posteriori error estimate for front-tracking approximate solutions to hyperbolic systems of conservation laws. The first section is dedicated to a proof of the global a posteriori error estimate while assuming certain local estimates. The remaining sections are dedicated to the proof of the required local estimates.

4.1 Main Results

Let u and v be two admissible front-tracking approximations of Lemma 2.2.2. Fix a time $t > 0$. Recall from Definition 2.1.2 that $\mathcal{D}\big(u(t)\big)$ is the set of all points of discontinuity in the approximation u at time t. As mentioned in section 2.2, at times when discontinuities meet the weights W_i decrease and therefore so does Φ. When no interactions occur, Φ is a differentiable function of time and we compute

$$\begin{aligned}\frac{d}{dt}\Big(\Phi\big(u(t),v(t)\big)\Big) &= \sum_{\substack{z\in\mathcal{D}(u(t))\bigcup\\ \mathcal{D}(v(t))}}\sum_{k=1}^{n}\Big\{\big|q_k(z-)\big|W_k(z-)-\big|q_k(z+)\big|W_k(z+)\Big\}\cdot\dot z\\ &= -\sum_{\substack{\delta\in\mathcal{D}(u(t))\bigcup\\ \mathcal{D}(v(t))}}\sum_{k=1}^{n}\Big[\;|q_k|W_k\,\dot z\;\Big]_z\,. \qquad (4.1.1)\end{aligned}$$

Recall the definition of the intermediate states $w_0(x) = u(t,x)$, and

$$w_k(x) = S_k\big(q_k(x)\big)\circ\cdots\circ S_1\big(q_1(x)\big)\big(u(t,x)\big).$$

Write

$$s_k(x) = s_k\big(w_k(x), w_{k-1}(x)\big), \qquad (4.1.2)$$

where $s_k(\cdot,\cdot)$ is the k-th eigenvalue of the matrix $A(\cdot,\cdot)$ defined in (1.1.9). For simplicity, we have suppressed the time dependence. Consider two neighboring discontinuities located at $z < y$. The approximations $u(t,\cdot)$ and $v(t,\cdot)$ are constant over the interval (z, y) and for every $x \in (z, y)$, we have

$$\big|q_k(z+)\big|s_k(z+)W_k(z+) = \big|q_k(x)\big|s_k(x)W_k(x) = \big|q_k(y-)\big|s_k(y-)W_k(y-). \tag{4.1.3}$$

For each interval between two neighboring discontinuities and for each k, add and subtract the terms in (4.1.3) to the sum (4.1.1). Then for each discontinuity, collect one term from each neighboring interval to obtain a contribution $\big[|q_k|s_kW_k\big]_z$. Since there exists a finite number of discontinuities, this leaves two terms unaccounted for over the intervals extending respectively to positive and negative infinity. Over these two intervals, $q_k(x)$ must be identically zero since the difference of the approximations $u(t)-v(t)$ belong to $L^1(\mathbb{R},\Omega)$. The two remaining terms are therefore zero. Following this procedure, (4.1.1) can be rewritten as

$$\begin{aligned}\frac{d}{dt}\Big(\Phi\big(u(t),v(t)\big)\Big) &= \sum_{z\in\mathcal{D}(u(t))\bigcup\mathcal{D}(v(t))}\sum_{k=1}^{n}\big[\;|q_k|W_k(s_k-\dot{z})\;\big]_z \\ &= \sum_{z\in\mathcal{D}(u(t))\bigcup\mathcal{D}(v(t))}\sum_{k=1}^{n}E_k(z)\end{aligned} \tag{4.1.4}$$

where we have introduced the terms

$$E_k(z) = \big|q_k(z+)\big|W_k(z+)\big(s_k(z+)-\dot{z}\big) - \big|q_k(z-)\big|W_k(z-)\big(s_k(z-)-\dot{z}\big). \tag{4.1.5}$$

We now present upper bounds for the local quantities $E_k(z)$ which can be computed a posteriori. In the following, $\mathcal{O}(1)$ will denote a quantity whose absolute value satisfies a uniform bound depending only on the system (1.1.2) and Ω.

Lemma 4.1.1 *Consider the strictly hyperbolic system* (1.1.2) *with genuinely nonlinear or linearly degenerate families in a neighborhood Ω of the origin. There exist positive constants $\delta_0, \kappa_1, \kappa_2$, depending only on the system* (1.1.2) *and Ω, with the following property. For any pair of front-tracking approximations u and v satisfying*

$$||u(0)||_{\mathrm{TV}},\ ||v(0)||_{\mathrm{TV}} < \delta_0,\ \textit{and}$$

$$u(0),\ v(0) \in L^1(\mathbb{R},\Omega), \tag{4.1.6}$$

and any position z and time t, we have

$$\sum_{k=1}^{n} E_k(z) \le \mathcal{O}(1) \sum_{\{\alpha | x_\alpha = z\}} D_\alpha(t). \tag{4.1.7}$$

In contrast to [8], where the local estimate for ϵ-approximations is of order $\mathcal{O}(1)\epsilon|\sigma_\alpha|$, we explicitly identify the approximation error and relate it to the residuals of u and v. Our global a posteriori error estimate is an immediate consequence of this local estimate.

Theorem 4.1.1 *Under the hypothesis of Lemma 4.1.1, there exists a constant C, depending only on the system* (1.1.2) *and on Ω, such that at any time t*

$$\Phi\big(u(t), v(t)\big) \le \Phi\big(u(0), v(0)\big) + C \int_0^t \Big[\sum_{\substack{\alpha \in \mathcal{W}(u(s)) \bigcup \\ \mathcal{W}(v(s))}} D_\alpha(s) \Big] \, ds. \tag{4.1.8}$$

Proof The coefficient $\mathcal{O}(1)$ in (4.1.7) is bounded over all waves by some constant C depending only on (1.1.2). Replace $\mathcal{O}(1)$ by C in inequality (4.1.7). Use (4.1.7) in equation (4.1.4) and integrate over time to obtain the final result. ■

Theorem 4.1.1 can be translated into a bound in the L^1 norm for the difference between an approximate solution and a weak solution.

Theorem 4.1.2 *Consider the strictly hyperbolic system* (1.1.2) *with genuinely nonlinear or linearly degenerate families in a neighborhood Ω of the origin. There exists positive constants δ_0 and C, depending only on* (1.1.2) *and Ω, with the following property: If u is any front-tracking approximation with $u(0) \in L^1(\mathbb{R}, \Omega)$ and v is any weak solution in $L^\infty\big([0,t], L^1(\mathbb{R}, \Omega)\big)$ which is the limit in L^1 of a sequence of ϵ-approximations and moreover, if both u and v satisfy*

$$||u(0)||_{\mathrm{TV}},\ ||v(0)||_{\mathrm{TV}} < \delta_0, \tag{4.1.9}$$

then we have the estimate

$$||u(t) - v(t)||_{L^1} \le C||u(0) - v(0)||_{L^1} + C \int_0^t \Big[\sum_{z \in \mathcal{D}(u(s))} D_z^*(s) \Big] \, ds. \tag{4.1.10}$$

Proof Let δ_0 be the constant given in Lemma 4.1.1. Let $\{v_\epsilon\}$ be the sequence of ϵ-approximations which converge in L^1 to the weak solution v. Since $||v(0)||_{\mathrm{TV}} <$

δ_0, we can assume that the approximations also satisfy this total variation bound. According to Theorem 4.1.1, for all ϵ we have

$$\Phi\big(u(t), v_\epsilon(t)\big) \le \Phi\big(u(0), v_\epsilon(0)\big) + C \int_0^t \Big[\sum_{\substack{\alpha \in \mathcal{W}(u(s)) \bigcup \\ \mathcal{W}(v_\epsilon(s))}} D_\alpha(s) \Big] \, ds. \tag{4.1.11}$$

Recall the bound (2.2.2) and the definition of the positive constants c_i in (2.3.8). Using this in (4.1.11), we find

$$c_1 ||u(t) - v_\epsilon(t)||_{L^1} \le 2c_2 ||u(0) - v_\epsilon(0)||_{L^1} + c_2 C \int_0^t \Big[\sum_{\substack{z \in \mathcal{D}(u(s)) \bigcup \\ \mathcal{D}(v_\epsilon(s))}} D_z^*(s) \Big] \, ds. \tag{4.1.12}$$

To demonstrate that the sum of the discrepancies of v_ϵ vanish as $\epsilon \to 0$, recall the form of the upper estimate in (4.1.7) originally found in [8],

$$\mathcal{O}(1) \sum_{\alpha \in \mathcal{W}(v_\epsilon(s))} D_\alpha(s) = \mathcal{O}(1) \sum_{\alpha \in \mathcal{W}(v_\epsilon(s))} \epsilon |\sigma_\alpha| \le \mathcal{O}(1) \epsilon ||v_\epsilon(s)||_{\mathrm{TV}}.$$

Taking the limit of (4.1.12) as $\epsilon \to 0$, the ϵ-approximations v_ϵ converge to v and the discrepancies of v_ϵ vanish. Dividing by c_1 completes the result. ■

4.2 Interaction Estimates

In this section we present estimates for wave interactions in shock coordinates and some lemmas characterizing quadratic quantities appearing in the proof of Lemma 4.1.1. The interaction estimates given below characterize the additivity of shock coordinates and are based on the original quadratic estimates of Glimm [17].

Throughout Sections 4.2 - 4.6, we will assume that the discontinuities under discussion belong to the approximate solution v, omitting the similar arguments for those in u. Since the time t and the wave α will be fixed for most of these computations, whenever convenient, we will remove the subscripts and superscripts involving the wave α. In particular, our notation $q_i\big(x_\alpha(t)\pm, t\big) = q_i^{\alpha\pm}$ will be simplified to $q_i^\pm$.

We begin with a lemma from Bressan, Liu, and Yang [8] concerning wave interactions.

Lemma 4.2.1 *If α is an isolated shock wave then*

$$\begin{aligned}
&|q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha| + \sum_{i\neq k_\alpha} |q^+_i - q^-_i| \\
&\qquad = \mathcal{O}(1)\Big(|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha| + |q^-_{k_\alpha}|\big) + |\sigma_\alpha| \sum_{i\neq k_\alpha} |q^-_i|\Big) && (4.2.1)\\
&\qquad = \mathcal{O}(1)\Big(|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big) + |\sigma_\alpha| \sum_{i\neq k_\alpha} |q^+_i|\Big). && (4.2.2)
\end{aligned}$$

If α is an isolated rarefaction wave then

$$\begin{aligned}
&|q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha| + \sum_{i\neq k_\alpha} |q^+_i - q^-_i| \\
&\qquad = \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha| + |q^-_{k_\alpha}|\big) + |\sigma_\alpha| \sum_{i\neq k_\alpha} |q^-_i|\Big) && (4.2.3)\\
&\qquad = \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big) + |\sigma_\alpha| \sum_{i\neq k_\alpha} |q^+_i|\Big). && (4.2.4)
\end{aligned}$$

If α is an isolated linearly degenerate wave then

$$\begin{aligned}
|q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha| + \sum_{i\neq k_\alpha} |q^+_i - q^-_i| &= \mathcal{O}(1)|\sigma_\alpha| \sum_{i\neq k_\alpha} |q^-_i| && (4.2.5)\\
&= \mathcal{O}(1)|\sigma_\alpha| \sum_{i\neq k_\alpha} |q^+_i|. && (4.2.6)
\end{aligned}$$

The following is a slight improvement to the previous estimates and can be derived in the same manner.

Lemma 4.2.2 *If α is an isolated shock wave and $-\sigma_\alpha \geq q^-_{k_\alpha} \geq 0$ then*

$$\begin{aligned}
&|q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha| + \sum_{i\neq k_\alpha} |q^+_i - q^-_i| \\
&\qquad = \mathcal{O}(1)\Big(|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha| - |q^-_{k_\alpha}|\big) + |\sigma_\alpha| \sum_{i\neq k_\alpha} |q^-_i|\Big). \qquad (4.2.7)
\end{aligned}$$

If α is an isolated rarefaction wave and $\sigma_\alpha \geq -q^-_{k_\alpha} \geq 0$ then

$$
\begin{aligned}
&|q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha| + \sum_{i \neq k_\alpha} |q^+_i - q^-_i| \\
&\qquad = \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| - |q^+_{k_\alpha}|\big) + |\sigma_\alpha| \sum_{i \neq k_\alpha} |q^+_i|\Big).
\end{aligned} \tag{4.2.8}
$$

The next estimate follows directly from (4.2.1) of Lemma 4.2.1. This weak form of the interaction estimates will be used in Lemma 4.2.4.

Lemma 4.2.3 *For any states $u, v, w \in \Omega$, we have*

$$\sum_{i=1}^n \big|q_i(u,v)\big| \leq \mathcal{O}(1)\Big(\sum_{i=1}^n \big|q_i(u,w)\big| + \big|q_i(u,w)\big|\Big). \tag{4.2.9}$$

During the proof of Lemma 4.1.1 the possibly large value of κ_1 will be mitigated by other factors appearing in the expansions.

Lemma 4.2.4 *For any positive value of κ_1, there exists a positive constant δ_1, depending only on κ_1 and the system* (1.1.2)*, such that if u and v are admissible front-tracking approximations satisfying*

$$||u(0)||_{L^\infty} + ||v(0)||_{\text{TV}},\ ||v(0)||_{L^\infty} + ||v(0)||_{\text{TV}} < \delta_1, \tag{4.2.10}$$

then

$$
\begin{aligned}
\kappa_1 \sum_{i=1}^n |q_i(x)| \ &\leq \ 1, \qquad \forall x \in \mathbb{R}, && (4.2.11)\\
\kappa_1 |\sigma_\alpha| \ &\leq \ 1, \qquad \forall \alpha \in \mathcal{W}\big(v(t)\big). && (4.2.12)
\end{aligned}
$$

Proof Let δ and K be the positive constants of Lemma 2.1.1. Let δ_0 be the constant of Lemma 2.2.2, which depends only on the given κ_1 and the system (1.1.2). We begin by assuming that condition (4.2.10) holds for

$$\delta_1 \leq \min\{\delta, \delta_0\}. \tag{4.2.13}$$

Fix the time t, the position x, and abbreviate $u(t,x) = u$, $v(t,x) = v$. From

(4.2.9) and the equivalence of the p_i and q_i coordinates, we have

$$\begin{aligned}\kappa_1 \sum_{i=1}^{n} \big|q_i(x)\big| = \kappa_1 \sum_{i=1}^{n} \big|q_i(u,v)\big| &\le \kappa_1 \mathcal{O}(1) \bigg(\sum_{i=1}^{n} \big|q_i(u,0)\big| + \big|q_i(0,v)\big| \bigg) \\ &= \kappa_1 \mathcal{O}(1) \bigg(\sum_{i=1}^{n} \big|p_i(u,0)\big| + \big|p_i(0,v)\big| \bigg). \qquad (4.2.14)\end{aligned}$$

Notice that since both approximations belong to L^1, they must vanish at infinity. The distance in wave coordinates between u and 0 is therefore bounded by the total variation of the approximation between $-\infty$ and x. Introduce a constant C_1 bounding the $\mathcal{O}(1)$ coefficient in (4.2.14) and depending only on the system (1.1.2). Applying our first observation and using C_1, we obtain

$$\begin{aligned}&\kappa_1 \mathcal{O}(1) \bigg(\sum_{i=1}^{n} \big|p_i(u,0)\big| + \big|p_i(0,v)\big| \bigg) \\ &\qquad \le \kappa_1 C_1 \bigg(\sum_{i=1}^{n} \big|p_i(u,0)\big| + \big|p_i(0,v)\big| \bigg) \\ &\qquad \le \kappa_1 C_1 \Big(V\big(u(t)\big) + V\big(v(t)\big) \Big) \\ &\qquad \le \kappa_1 C_1 \bigg[\Big(V\big(u(t)\big) + V\big(v(t)\big) \Big) + K \Big(Q\big(u(t)\big) + Q\big(v(t)\big) \Big) \bigg]. \qquad (4.2.15)\end{aligned}$$

Note that there exists a constant C_2 such that $V(\cdot) \le C_2 ||\cdot||_{\mathrm{TV}}$ and that $V(\cdot) \le Q(\cdot)^2$. We now further restrict the value of δ_1 by setting

$$\delta_1 = \min \Big\{ \delta, \delta_0, \frac{1}{KC_2}, \frac{1}{2\kappa_1 C_2}, \frac{1}{4\kappa_1 C_1 C_2} \Big\}. \qquad (4.2.16)$$

According to Lemma 2.1.1, $V + KQ$ is a decreasing functional. Continuing from (4.2.15) and using the bounds (4.2.10) and (4.2.16), we conclude that

$$\begin{aligned}\kappa_1 \sum_{i=1}^{n} \big|q_i(x)\big| &\le \kappa_1 C_1 \bigg[\Big(V\big(u(t)\big) + V\big(v(t)\big) \Big) + K \Big(Q\big(u(t)\big) + Q\big(v(t)\big) \Big) \bigg] \\ &\le \kappa_1 C_1 \bigg[\Big(V\big(u(0)\big) + V\big(v(0)\big) \Big) + K \Big(Q\big(u(0)\big) + Q\big(v(0)\big) \Big) \bigg]\end{aligned}$$

$$
\begin{aligned}
&\le \kappa_1 C_1 \Big[\Big(V\big(u(0)\big) + V\big(v(0)\big) \Big) + K \Big(V\big(u(0)\big)^2 + V\big(v(0)\big) \Big)^2 \Big] \\
&\le \kappa_1 C_1 \Big[C_2 \Big(\big|\big|u(0)\big|\big|_{\mathrm{TV}} + K C_2 \big|\big|u(0)\big|\big|_{\mathrm{TV}}^2 \Big) \\
&\qquad + C_2 \Big(\big|\big|v(0)\big|\big|_{\mathrm{TV}} + K C_2 \big|\big|v(0)\big|\big|_{\mathrm{TV}}^2 \Big) \Big] \\
&\le 2\kappa_1 C_1 C_2 \Big[\big|\big|u(0)\big|\big|_{\mathrm{TV}} + \big|\big|v(0)\big|\big|_{\mathrm{TV}} \Big] \le \frac{1}{2} + \frac{1}{2} = 1. \qquad (4.2.17)
\end{aligned}
$$

To demonstrate (4.2.12), it suffices to notice that

$$
\begin{aligned}
\kappa_1 |\sigma_\alpha| &\le \kappa_1 V\big(v(t)\big) \\
&\le \kappa_1 \Big(V\big(v(0)\big) + K Q\big(v(0)\big) \Big) \\
&\le \kappa_1 \Big(\big|\big|v(0)\big|\big|_{\mathrm{TV}} + K C_2^2 \big|\big|v(0)\big|\big|_{\mathrm{TV}}^2 \Big) \\
&\le 2\kappa_1 C_2 \big|\big|v(0)\big|\big|_{\mathrm{TV}} \le 1. \qquad (4.2.18)
\end{aligned}
$$

■

The next three lemmas characterize similar expressions appearing in the proof of Lemma 4.1.1. The first estimate in Lemma 4.2.5 is taken from [8], with a slight improvement when the factors $\sigma + \sigma' = 0$. Lemmas 4.2.6 and 4.2.7 explicitly relate quadratic expressions in the proof of Lemma 4.1.1 to the discrepancy of shock and rarefaction waves.

Lemma 4.2.5 *Choose* $\overline{w} \in \Omega, \sigma, \sigma' \in \mathbb{R}, \quad k \in \{1, \dots, n\}$. *Define the states and the wave speeds*

$$
\begin{array}{rclrcl}
w &=& S_k(\sigma)(\overline{w}), & s &=& s_k(\overline{w}, w), \\
w' &=& S_k(\sigma')(w), & s' &=& s_k(w, w'), \\
w'' &=& S_k(\sigma + \sigma')(\overline{w}), & s'' &=& s_k(\overline{w}, w'').
\end{array}
$$

If the family k *is genuinely nonlinear then*

$$
\big|(\sigma + \sigma')(s'' - s') - \sigma(s - s')\big| = \mathcal{O}(1)|\sigma\sigma'| \cdot |\sigma + \sigma'|. \qquad (4.2.19)
$$

Proof Define the C^2 function

$$
Q(\sigma, \sigma') \equiv (\sigma + \sigma')s'' - \sigma s - \sigma' s' = (\sigma + \sigma')(s'' - s') - \sigma(s - s').
$$

Notice that

$$Q(\sigma,0)=0 \qquad \text{and} \qquad Q(0,\sigma')=0, \qquad \forall \sigma,\sigma'.$$

Only mixed derivatives remain in Taylor's expansion of Q. We compute the first mixed derivative

$$\frac{\partial Q}{\partial \sigma'} = s'' + (\sigma+\sigma')\frac{\partial s''}{\partial \sigma'} - s' - \sigma'\frac{\partial s'}{\partial \sigma'}, \tag{4.2.20}$$

$$\frac{\partial^2 Q}{\partial \sigma \partial \sigma'} = \frac{\partial s''}{\partial \sigma} + \frac{\partial s''}{\partial \sigma'} + (\sigma+\sigma')\frac{\partial^2 s''}{\partial \sigma \partial \sigma'} - \frac{\partial s'}{\partial \sigma} - \sigma'\frac{\partial^2 s'}{\partial \sigma \partial \sigma'}. \tag{4.2.21}$$

Using the identity (1.1.15) and the fact that to second order the shock speed is the average of the characteristic speeds of its neighboring states, we find

$$\begin{aligned}
\frac{\partial s''}{\partial \sigma} &= \frac{\partial}{\partial \sigma}\Big[s_{k_\alpha}\big(\overline{w}, S_{k_\alpha}(\sigma+\sigma')(\overline{w})\big)\Big] \\
&= \frac{\partial}{\partial \sigma}\left[\frac{1}{2}\Big(\lambda_{k_\alpha}(\overline{w}) + \lambda_{k_\alpha}\big(S_{k_\alpha}(\sigma+\sigma')(\overline{w})\big)\Big) + \mathcal{O}(1)|\sigma+\sigma'|\right] \\
&= \frac{1}{2} + \frac{\partial}{\partial \sigma}\big(\mathcal{O}(1)|\sigma+\sigma'|^2\big).
\end{aligned} \tag{4.2.22}$$

Evaluating (4.2.22) at the origin,

$$\frac{\partial s''}{\partial \sigma}(0,0) = \frac{1}{2}. \tag{4.2.23}$$

Similarly,

$$\frac{\partial s''}{\partial \sigma'}(0,0) = \frac{1}{2}. \tag{4.2.24}$$

When $\sigma'=0$, then $w' = S_{k_\alpha}(\sigma')(w) = w$. Combining this fact with the parameterization (1.1.15), we verify that

$$\begin{aligned}
\frac{\partial s'}{\partial \sigma}(0,0) &= \frac{d}{d\sigma}\Big[s_{k_\alpha}(w,w')\big|_{\sigma'=0}\Big] \\
&= \frac{d}{d\sigma}\Big[s_{k_\alpha}(w,w)\Big] \\
&= \frac{d}{d\sigma}\Big[\lambda_{k_\alpha}\big(S_{k_\alpha}(\sigma)(\overline{w})\big)\Big] = 1.
\end{aligned} \tag{4.2.25}$$

Evaluating the mixed derivative (4.2.21) at the origin, identities (4.2.23), (4.2.24), and (4.2.25) imply

$$\frac{\partial^2 Q}{\partial\sigma\partial\sigma'} = 0.$$

Integrate this derivative to compute

$$Q(\sigma,\sigma') = \int_0^{\sigma}\int_0^{\sigma'} \frac{\partial^2 Q}{\partial\sigma\partial\sigma'}(r,s)\ drds = \mathcal{O}(1)\int_0^{|\sigma|}\int_0^{|\sigma'|} |rs|\ drds$$

$$= \mathcal{O}(1)|\sigma\sigma'|\big(|\sigma| + |\sigma'|\big).$$

In fact, when $\sigma + \sigma' = 0$ then $s - s' = 0$ and Q vanishes. Q is therefore

$$Q(\sigma,\sigma') = \mathcal{O}(1)|\sigma\sigma'|\cdot|\sigma + \sigma'|.$$

■

We now introduce two quadratic quantities, similar to (4.2.19), associated with shock and rarefaction waves. First, choose a shock wave α, a state $v^- \in \Omega$, and define

$$\widehat{v} = S_{k_\alpha}(-q)(v^-), \text{ and } v^* = S_{k_\alpha}(q + \sigma_\alpha)(\widehat{v}).$$

The quadratic shock remainder is the function

$$Q_{\text{shk}}(q,s) \equiv -(q + \sigma_\alpha)\big(s_{k_\alpha}(\widehat{v}, v^*) - s\big) - q\big(s_{k_\alpha}(v^-,\widehat{v}) - s\big), \tag{4.2.26}$$

for $(q,s) \in [0, |\sigma_\alpha|] \times \mathbb{R}$. If α is a rarefaction wave, v^+ is some fixed state of Ω, and

$$\overline{v} = R_{k_\alpha}(-q)(v^+), \qquad v^- = R_{k_\alpha}(-\sigma_\alpha)(v^+),$$

then we define the quadratic rarefaction remainder

$$\begin{aligned} Q_{\text{rar}}(q,s) \equiv & q\Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v) + \lambda_{k_\alpha}(\overline{v})\big) - s\Big) \\ & - (\sigma_\alpha - q)\Big(\frac{1}{2}\big(\lambda_{k_\alpha}(\overline{v}) + \lambda_{k_\alpha}(v^-)\big) - s\Big), \end{aligned} \tag{4.2.27}$$

for $(q,s) \in [0,\sigma_\alpha] \times \mathbb{R}$. Notwithstanding the weight W_{k_α}, these remainders and (4.2.19) are the contribution from the k_α-th coordinate to the term $E_{k_\alpha}(\alpha)$.

Lemma 4.2.6 *There exists a positive constant δ_2, depending only on the system*

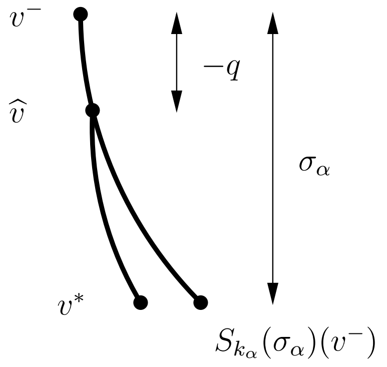

Figure 4.1: Illustration of the states defining Q_{shk}.

(1.1.2)*, such that if* $|\sigma_\alpha| < \delta_2$ *then*

$$\sup_{q\in[0,|\sigma_\alpha|]} Q_{shk}(q, \dot{x}_\alpha) = D_\alpha(t). \tag{4.2.28}$$

Proof Figure 4.1 illustrates the states in the construction of Q_{shk}. Introduce the function

$$M(q) = \frac{1}{2}(q+\sigma_\alpha)s_{k_\alpha}(\widehat{v}, v^*) + \frac{q}{2}s_{k_\alpha}(v^-, \widehat{v}) \tag{4.2.29}$$

for which

$$Q_{\text{shk}}(q, s) = 2\Big[(q + \frac{1}{2}\sigma_\alpha)s - M(q)\Big]. \tag{4.2.30}$$

We begin by demonstrating that M is a concave function of q over the interval $\big[0, |\sigma_\alpha|\big]$. The parameterization (1.1.15) along shock curves provides two facts: first

$$\frac{d}{dq}\Big(\lambda_{k_\alpha}\big(\widehat{v}(q)\big)\Big) = \frac{d}{dq}\Big(\lambda_{k_\alpha}\big(S_{k_\alpha}(-q)(v^-)\big)\Big) = -1; \tag{4.2.31}$$

and secondly,

$$\begin{aligned} \lambda_{k_\alpha}(\widehat{v}) - \lambda_{k_\alpha}(v^-) &= -q, & (4.2.32)\\ \lambda_{k_\alpha}(v^*) - \lambda_{k_\alpha}(\widehat{v}) &= q + \sigma_\alpha, & (4.2.33)\\ \lambda_{k_\alpha}(v^+) - \lambda_{k_\alpha}(v^-) &= \sigma_\alpha. & (4.2.34) \end{aligned}$$

Adding equation (4.2.32) to (4.2.33) and then comparing the sum to (4.2.34), we conclude that

$$\lambda_{k_\alpha}(v^+) = \lambda_{k_\alpha}(v^*). \tag{4.2.35}$$

Using (4.2.35) and the fact that the Rankine-Hugoniot speeds are approximately the average of characteristic speeds of the end states, we obtain

$$\begin{aligned} M(q) = \frac{1}{4}(q+\sigma_\alpha)\Big(\lambda_{k_\alpha}(\widehat{v}) + \lambda_{k_\alpha}(v^+)\Big) + \mathcal{O}(1)(q+\sigma_\alpha)^3 \\ + \frac{q}{4}\Big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(\widehat{v})\Big) + \mathcal{O}(1)q^3. \end{aligned} \tag{4.2.36}$$

Concentrating on the lower order terms, and using (4.2.31), (4.2.32), and (4.2.33), we have

$$\begin{aligned} &\frac{d^2}{dq^2}\left[\frac{1}{4}(q+\sigma_\alpha)\Big(\lambda_{k_\alpha}(\widehat{v}) + \lambda_{k_\alpha}(v^+)\Big) + \frac{q}{4}\Big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(\widehat{v})\Big)\right] \\ &\quad = \frac{d}{dq}\left[\frac{1}{4}\big(\lambda_{k_\alpha}(v^-) + 2\lambda_{k_\alpha}(\widehat{v}) + \lambda_{k_\alpha}(v^+)\big) - \frac{1}{4}(q+\sigma_\alpha) - \frac{q}{4}\right] \\ &\quad = -1. \end{aligned}$$

Since the higher order terms are cubic in q and $q+\sigma_\alpha$, both of which are bounded by $|\sigma_\alpha|$ for $|\sigma_\alpha|$ sufficiently small, the concavity of M results from the concavity of the lower order terms. Notice that the bound on $|\sigma_\alpha|$ depends entirely on the $\mathcal{O}(1)$ coefficient which has upper bounds computable from the system (1.1.2). Thus as asserted, M is concave when $|\sigma_\alpha| \le \mathcal{O}(1)$.

The derivative of Q_{shk} computed from identity (4.2.30) is

$$\frac{dQ_{\text{shk}}}{dq} = 2\Big[s - \frac{dM}{dq}\Big]. \tag{4.2.37}$$

If M is concave then M' is monotonic decreasing over $\big[0, |\sigma_\alpha|\big]$. We complete the proof by considering three cases. First suppose $s \in \big[M'\big(|\sigma_\alpha|\big), M'(0)\big]$. Then by (4.2.37) the function $Q_{\text{shk}}(\cdot, s)$ has a critical point. This critical point must be a global minimum in $q \in \big[0, |\sigma_\alpha|\big]$ and the maximum of the function $Q_{\text{shk}}(\cdot, s)$ must occur at one of the boundaries. This situation is illustrated in Figure 4.2. Note that $\widehat{v}(0) = v^-$, $v^*(0) = v^+$, while $\widehat{v}\big(|\sigma_\alpha|\big) = v^+$. Using these identities to evaluate $Q_{\text{shk}}(\cdot, s)$ at the

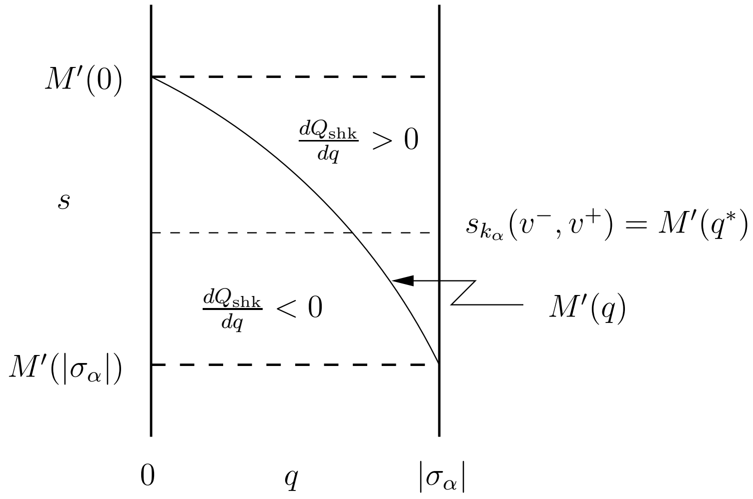
$M'(0)$
$\frac{dQ_{\text{shk}}}{dq} > 0$
s
$s_{k_\alpha}(v^-, v^+) = M'(q^*)$
$\frac{dQ_{\text{shk}}}{dq} < 0$
$M'(q)$
$M'(|\sigma_\alpha|)$
0
q
$|\sigma_\alpha|$

Figure 4.2: Behavior of Q_{shk} over the domain $\big[0, |\sigma_\alpha|\big] \times \mathbb{R}$.

boundary, we find that

$$\begin{aligned}\sup_{q\in[0,|\sigma_\alpha|]} Q_{\text{shk}}(q,s) &= \max\Big\{Q_s(0,s), Q_{\text{shk}}(|\sigma_\alpha|, s)\Big\}\\ &= \max\Big\{\sigma_\alpha\big(s - s_{k_\alpha}(v^-,v^+)\big), |\sigma_\alpha|\big(s - s_{k_\alpha}(v^-,v^+)\big)\Big\}\\ &= |\sigma_\alpha|\big|s - s_{k_\alpha}(v^-,v^+)\big|.\end{aligned}$$

This quantity is equal to $D_\alpha(t)$ when $s = \dot{x}_\alpha$.

When $s \in \big(M'(0), +\infty\big)$, equation (4.2.37) shows that the supremum of $Q_{\text{shk}}(\cdot, s)$ occurs at the boundary $q = |\sigma_\alpha|$. Computing this, we obtain

$$\sup_{q\in[0,|\sigma_\alpha|]} Q_{\text{shk}}(q,s) = Q_{\text{shk}}(|\sigma_\alpha|, s) = |\sigma_\alpha|\big(s - s_{k_\alpha}(v^-,v^+)\big). \tag{4.2.38}$$

We need to show that this quantity is in fact positive and therefore equal to $D_\alpha(t)$ when $s = \dot{x}_\alpha$. According to the mean value theorem, there exists $q^* \in \big(0, |\sigma_\alpha|\big)$ such that

$$\begin{aligned}M'(q^*) &= \frac{1}{|\sigma_\alpha|}\Big(M\big(|\sigma_\alpha|\big) - M(0)\Big)\\ &= \frac{1}{|\sigma_\alpha|}\Big(\frac{|\sigma_\alpha|}{2} - \frac{\sigma_\alpha}{2}\Big)s_{k_\alpha}(v^-, v^+)\\ &= s_{k_\alpha}(v^-, v^+).\end{aligned}$$

Since M' is monotonic decreasing, $s_{k_\alpha}(v^-, v^+) = M'(q^*) < M'(0) \le s$ and therefore the supremum in (4.2.38) is positive and equal to the discrepancy. With a similar argument, we verify the lemma when $s \in \big(-\infty, M'(|\sigma_\alpha|)\big)$. This completes the proof. ■

Lemma 4.2.7 *With the definition* (4.2.27), *we have*

$$\sup_{q\in[0,\sigma_\alpha]} Q_{rar}(q, \dot{x}_\alpha) = D_\alpha(t). \tag{4.2.39}$$

Proof To compute the maximum of Q_{rar} we begin by examining its derivative. The parameterization (1.1.15) along rarefaction curves and the definition of $\overline{v}$ imply that

$$\frac{d}{dq}\Big(\lambda_{k_\alpha}(\overline{v})\Big) = \frac{d}{dq}\Big(\lambda_{k_\alpha}\big(R_{k_\alpha}(-q)(v^+)\big)\Big) = -1.$$

Using this fact, we proceed to compute

$$\begin{aligned}\frac{\partial Q_{\text{rar}}}{\partial q} =&\Big[\frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(\overline{v})\big) - s\Big] + q\Big(-\frac{1}{2}\Big)\\ &+ \Big[\frac{1}{2}\big(\lambda_{k_\alpha}(\overline{v}) + \lambda_{k_\alpha}(v^-)\big) - s\Big] - (\sigma_\alpha - q)\Big(-\frac{1}{2}\Big)\\ =&2\Big[\frac{1}{4}\big(\lambda_{k_\alpha}(v^+) + 2\lambda_{k_\alpha}(\overline{v}) + \lambda_{k_\alpha}(v^-)\big) + \frac{1}{2}\big(\frac{1}{2}\sigma_\alpha - q\big) - s\Big].\end{aligned} \tag{4.2.40}$$

Taking another derivative gives

$$\frac{\partial^2 Q_{\text{rar}}}{\partial q^2} = -2. \tag{4.2.41}$$

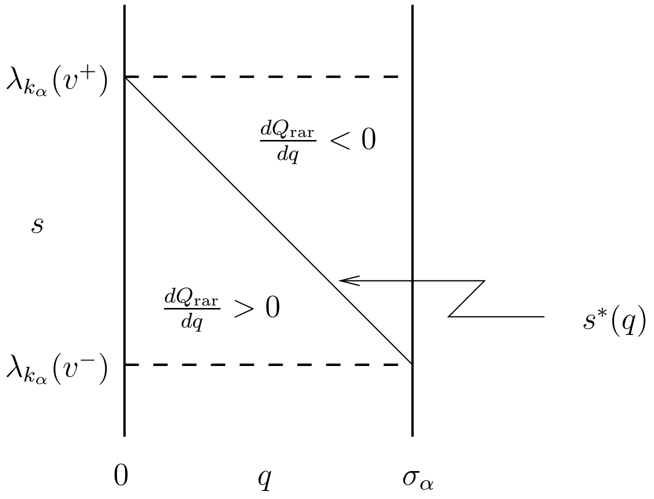
$\lambda_{k_\alpha}(v^+)$
$\frac{dQ_{\text{rar}}}{dq} < 0$
s
$\frac{dQ_{\text{rar}}}{dq} > 0$
$s^*(q)$
$\lambda_{k_\alpha}(v^-)$
0
q
σ_α

Figure 4.3: Behavior of Q_{rar} over the domain $\big[0, \sigma_\alpha\big] \times \mathbb{R}$

With the help of the identity $\lambda_{k_\alpha}(v^+) - \lambda_{k_\alpha}(v^-) = \sigma_\alpha$, we compute

$$\begin{aligned}\frac{\partial Q_{\text{rar}}}{\partial q}(0) &= \frac{1}{2}\Big(\lambda_{k_\alpha}(v^+) + 2\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(v^-)\Big) + \frac{1}{2}\sigma_\alpha - 2s \\ &= \frac{1}{2}\Big(\lambda_{k_\alpha}(v^+) + 2\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(v^+) - \sigma_\alpha\Big) + \frac{1}{2}\sigma_\alpha - 2s \\ &= 2\lambda_{k_\alpha}(v^+) - 2s. \qquad (4.2.42)\end{aligned}$$

From (4.2.41) and (4.2.42), we conclude

$$\frac{\partial Q_{\text{rar}}}{\partial q} = 2\lambda_{k_\alpha}(v^+) - 2s - 2q. \qquad (4.2.43)$$

We need to find the supremum of Q_{rar} for all $q \in [0, \sigma_\alpha]$ and all $s \in \mathbb{R}$. Formula (4.2.43) shows that

$$\frac{\partial Q_{\text{rar}}}{\partial q} > 0 \text{ when } s < \lambda_{k_\alpha}(v^+) - q, \qquad (4.2.44)$$

and

$$\frac{\partial Q_{\text{rar}}}{\partial q} < 0 \text{ when } \lambda_{k_\alpha}(v^+) - q < s. \qquad (4.2.45)$$

Since q is contained in $[0, \sigma_\alpha]$, we have

$$\lambda_{k_\alpha}(v^-) \le \lambda_{k_\alpha}(v^+) - q \le \lambda_{k_\alpha}(v^+). \tag{4.2.46}$$

We divide the remainder of the proof into three cases. Refer to Figure 4.3 for a representation of these regions in the (q, s) plane. If $\lambda_{k_\alpha}(v^+) < s$ then (4.2.45) and (4.2.46) show that the maximum of $Q_{\text{rar}}(\cdot, s)$ occurs when $q = 0$. For $s = \dot{x}_\alpha$, this maximum is

$$\begin{aligned}\sup_{q\in[0,\sigma_\alpha]} Q_{\text{rar}}(q, \dot{x}_\alpha) = Q_{\text{rar}}(0, \dot{x}_\alpha) &= -\sigma_\alpha \Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(v^-)\big) - \dot{x}_\alpha\Big) \\ &= D_\alpha(t).\end{aligned}$$

If $s = \dot{x}_\alpha < \lambda_{k_\alpha}(v^-)$, then (4.2.44) and (4.2.46) imply that

$$\begin{aligned}\sup_{q\in[0,\sigma_\alpha]} Q_{\text{rar}}(q, \dot{x}_\alpha) = Q_{\text{rar}}(\sigma_\alpha, \dot{x}_\alpha) &= \sigma_\alpha \Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(v^-)\big) - \dot{x}_\alpha\Big) \\ &= D_\alpha(t).\end{aligned}$$

If $s \in I \equiv \big[\lambda_{k_\alpha}(v^-), \lambda_{k_\alpha}(v^+)\big]$, then the supremum of $Q_{\text{rar}}(\cdot, s)$ occurs when

$$\lambda_{k_\alpha}(v^+) - q = s.$$

Let $s^*(q) \equiv \lambda_{k_\alpha}(v^+) - q$. Notice that s^* parameterizes the interval I as q traverses $[0, \sigma_\alpha]$. We can therefore say that, for $(q, s) \in [0, \sigma_\alpha] \times I$, the supremum of Q_{rar} occurs along the diagonal $\big(q, s^*(q)\big)$. Using identity (4.2.43), the derivative along the diagonal is

$$\frac{d}{dq}\Big(Q_{\text{rar}}\big(q, s^*(q)\big)\Big) = 2(q - \frac{1}{2}\sigma_\alpha). \tag{4.2.47}$$

We also verify that

$$\begin{aligned}Q_{\text{rar}}\big(0, s^*(0)\big) &= -\sigma_\alpha\Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(v^-)\big) - \lambda_{k_\alpha}(v^+)\Big) \\ &= \frac{\sigma_\alpha^2}{2}.\end{aligned} \tag{4.2.48}$$

Consider now the discrepancy of the rarefaction wave α for $\dot{x}_\alpha \in I$, which in

Lemma 2.3.1 was found to be

$$D_\alpha(t) = \frac{1}{2}\big(\lambda_{k_\alpha}(v_\alpha^-) - \dot{x}_\alpha\big)^2 + \frac{1}{2}\big(\dot{x}_\alpha - \lambda_{k_\alpha}(v_\alpha^+)\big)^2. \tag{4.2.49}$$

Parameterizing the $\dot{x}_\alpha \in I$ by $s^*(q)$ and using (4.2.49), we have

$$\begin{aligned}
\frac{d}{dq}D_\alpha(t) &= \frac{d}{ds^*}\Big(D_\alpha(t)\Big)\frac{ds^*}{dq} \\
&= \frac{d}{ds^*}\Big[\frac{1}{2}\big(\lambda_{k_\alpha}(v_\alpha^-) - s^*\big)^2 + \frac{1}{2}\big(s^* - \lambda_{k_\alpha}(v_\alpha^+)\big)^2\Big](-1) \\
&= \Big[(\lambda_{k_\alpha}(v^-) - \lambda_{k_\alpha}(v^+) + q) + (\lambda_{k_\alpha}(v^+) - \lambda_{k_\alpha}(v^+) + q)\Big] \\
&= 2(q - \frac{1}{2}\sigma_\alpha).
\end{aligned} \tag{4.2.50}$$

Finally, when $\dot{x}_\alpha = s^*(0)$

$$D_\alpha(t) = \frac{\sigma_\alpha^2}{2}. \tag{4.2.51}$$

As a function of q, $Q_{\text{rar}}\big(q, s^*(q)\big)$ and $D_\alpha(t)$ agree when $q = 0$, their derivatives agree throughout $[0, \sigma_\alpha]$, and therefore they must be equal. ■

4.3 Isolated Waves: $E_k(\alpha)$ When $k \neq k_\alpha$

In this section we demonstrate a preliminary estimate for the terms $E_k(\alpha)$ defined in (4.1.5). We assume that the waves are isolated and belong to the approximation v. In this case, their left and right hand states, as defined in 2.1.3, coincide with the left and right sided limits of v at $\dot{x}_\alpha$, denoted v^- and v^+. Lemmas 4.2.4 and 4.2.6 introduced two upper bounds on the total variation, δ_1 and δ_2. We therefore combine these bounds with the admissibility requirements of Lemma 2.2.2 and assume throughout this section that our approximations satisfy

$$||u(0)||_{L^\infty} + ||u(0)||_{\text{TV}},\ ||v(0)||_{L^\infty} + ||v(0)||_{\text{TV}} < \min\big\{\delta_0, \delta_1, \delta_2\big\}. \tag{4.3.1}$$

In Section 4.6, where non-isolated waves will be examined, traveling discontinuities contain n waves only one of which travels at a speed satisfying condition (2.1.5), $\dot{x}_\alpha \in \lambda_{k_\alpha}(\Omega)$. In anticipation of this fact, we present the results of this section for isolated waves α whose wave speeds are only required to satisfy

$$|\dot{x}_\alpha| \leq \Lambda, \tag{4.3.2}$$

for the constant Λ of (2.1.14). Recall the definition (1.1.24) of the lower bound d on wave speed separation and on genuine nonlinearity.

Lemma 4.3.1 *For any isolated wave $\alpha \in \mathcal{W}\big(v(t)\big)$, any $k \neq k_\alpha$, and either sign $s \in \{+,-\}$, we have*

$$\begin{aligned} E_k(\alpha) \leq & -\kappa_1 d|\sigma_\alpha||q_k^s| + \mathcal{O}(1)D_\alpha(t) \\ & + \mathcal{O}(1)\Big(|\sigma_\alpha||q_{k_\alpha}^s|\big(|\sigma_\alpha| + |q_{k_\alpha}^s|\big) + |\sigma_\alpha|\sum_{i\neq k_\alpha}|q_i^s|\Big). \end{aligned} \tag{4.3.3}$$

If the isolated wave satisfies $q_{k_\alpha}^- q_{k_\alpha}^+ < 0$ then there exists a sign s for which

$$\begin{aligned} E_k(\alpha) \leq & -\kappa_1 d|\sigma_\alpha||q_k^s| + \mathcal{O}(1)D_\alpha(t) \\ & + \mathcal{O}(1)\Big(|\sigma_\alpha||q_{k_\alpha}^s|\big(|\sigma_\alpha| - |q_{k_\alpha}^s|\big) + |\sigma_\alpha|\sum_{i\neq k_\alpha}|q_i^s|\Big). \end{aligned} \tag{4.3.4}$$

If the wave belongs to a linearly degenerate family then,

$$E_k(\alpha) \leq -\kappa_1 d|\sigma_\alpha||q_k^-| + D_\alpha(t) + \mathcal{O}(1)|\sigma_\alpha|\sum_{i\neq k_\alpha}|q_i^-|. \tag{4.3.5}$$

For these estimates, the wave speed $\dot{x}_\alpha$ is only required to satisfy (4.3.2).

Proof We begin with the proof of equation (4.3.3) for a shock wave α and s positive. The other cases are similar and will be discussed later. Since the wave is isolated, its left and right hand states are the neighboring states of the discontinuity v^- and v^+. Rewrite the contribution of the wave α to the derivative of Φ as

$$\begin{aligned} E_k(\alpha) =& |q_k^+|W_k^+(s_k^+ - \dot{x}_\alpha) - |q_k^-|W_k^-(s_k^- - \dot{x}_\alpha) \\ =& |q_k^+|W_k^+(s_k^+ - s_k^-) + \big(|q_k^+| - |q_k^-|\big)W_k^-(s_k^- - \dot{x}_\alpha) \\ & + |q_k^+|(W_k^+ - W_k^-)(s_k^- - \dot{x}_\alpha). \end{aligned} \tag{4.3.6}$$

When $k \neq k_\alpha$, the first term is handled by combining the bound $W_k^+ \leq 2$ with the simple bound $|s_k^+ - s_k^-| = \mathcal{O}(1)|\sigma_\alpha|$. This gives

$$|q_k^+|W_k^+(s_k^+ - s_k^-) = \mathcal{O}(1)|q_k^+||\sigma_\alpha|. \tag{4.3.7}$$

Since the absolute value of the speed $\dot{x}_\alpha$ is bounded by Λ which can be determined entirely from the system, the quantity $(s_{k_\alpha}^- - \dot{x}_\alpha)$ is $\mathcal{O}(1)$. Combining this with the bound (4.2.2) of Lemma 4.2.1, we obtain an estimate for the second term in (4.3.6).

$$\begin{aligned} \big(|q_k^+| - |q_k^-|\big)W_k^-(s_k^- - \dot{x}_\alpha) \leq & \mathcal{O}(1)|q_k^+ - q_k^-| \\ \leq & \mathcal{O}(1)\Big(|\sigma_\alpha||q_{k_\alpha}^+|\big(|\sigma_\alpha| + |q_{k_\alpha}^+|\big) + |\sigma_\alpha|\sum_{i\neq k_\alpha}|q_i^+|\Big). \end{aligned} \tag{4.3.8}$$

The third term of (4.3.6) remains to be bounded. We rewrite it as

$$\begin{aligned}|q_k^+|(W_k^+ - W_k^-)(s_k^- - \dot{x}_\alpha) =&|q_k^+|(W_k^+ - W_k^-)\big(s_k^- - s_{k_\alpha}(v^-, v^+)\big)\\ &+ |q_k^+|(W_k^+ - W_k^-)\big(s_{k_\alpha}(v^-, v^+) - \dot{x}_\alpha\big).\end{aligned} \tag{4.3.9}$$

When $k_\alpha > k$, equations (1.1.25) and (2.2.6) imply that

$$s_k^- + d < s_{k_\alpha}(v^-, v^+) \qquad \text{and} \qquad W_k^+ - W_k^- = \kappa_1 |\sigma_\alpha| \ ,$$

while for $k_\alpha < k$, they give

$$s_{k_\alpha}(v^-, v^+) + d < s_k^- \qquad \text{and} \qquad W_k^+ - W_k^- = -\kappa_1 |\sigma_\alpha| \ .$$

For both choices of k, the first term in (4.3.9) is negative. Using the estimate of Lemma 4.2.4, $\kappa_1 |q_k^+| \le 1$, (2.2.6), and (2.3.11) shows that the second term is bounded by the discrepancy. The last term in (4.3.6) therefore satisfies

$$|q_k^+|(W_k^+ - W_k^-)(s_k^- - \dot{x}_\alpha) \le -\kappa_1 d |\sigma_\alpha||q_k^+| + D_\alpha(t). \tag{4.3.10}$$

To obtain the final estimate for $k \neq k_\alpha$, sum (4.3.7), (4.3.8), and (4.3.10). We have shown that

$$\begin{aligned}E_k(\alpha) \le& -\kappa_1 d|\sigma_\alpha||q_k^+| + D_\alpha(t)\\ &+ \mathcal{O}(1)\Big(|\sigma_\alpha||q_{k_\alpha}^+|\big(|\sigma_\alpha| + |q_{k_\alpha}^+|\big) + |\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^+|\Big).\end{aligned} \tag{4.3.11}$$

To demonstrate the result for a rarefaction wave, still with s positive, follow the same procedure as for shock waves. Estimates (4.3.7) and (4.3.10) remain the same. When estimating the second term in (4.3.6), use equation (4.2.4) of Lemma 4.2.1 to obtain the following replacement to (4.3.8):

$$\begin{aligned}(|q_k^+| - |q_k^-|)W_k^-(s_k^- - \dot{x}_\alpha) \le& \mathcal{O}(1)|q_k^+ - q_k^-|\\ \le& \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q_{k_\alpha}^+|(|\sigma_\alpha| + |q_{k_\alpha}^+|)\\ &+ |\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^+|\Big).\end{aligned} \tag{4.3.12}$$

Summing (4.3.7), (4.3.9), and (4.3.12) leads to the introduction of a cubic term $|\sigma_\alpha|^3$ in the estimate

$$\begin{aligned}E_k(\alpha) \le& -\kappa_1 d|\sigma_\alpha||q_k^+| + D_\alpha(t)\\ &+ \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q_{k_\alpha}^+|\big(|\sigma_\alpha| + |q_{k_\alpha}^+|\big) + |\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^+|\Big).\end{aligned} \tag{4.3.13}$$

To obtain the form (4.3.3) of the result, we show how to control the cubic term with the discrepancy. In Lemma 2.3.2 we showed that the infimum of $D_\alpha(t)$ was $\sigma_\alpha^2/2$. Using this we have

$$\begin{aligned}\mathcal{O}(1)|\sigma_\alpha|^3 &= \left[\frac{\mathcal{O}(1)|\sigma_\alpha|^3}{D_\alpha(t)}\right] D_\alpha(t)\\ &\leq \mathcal{O}(1)|\sigma_\alpha| D_\alpha(t) = \mathcal{O}(1) D_\alpha(t). \end{aligned} \tag{4.3.14}$$

Using this in (4.3.13) proves (4.3.3).

We now outline the argument for a linearly degenerate wave α when s is positive. Starting from expansion (4.3.6), we can obtain for the first term an estimate identical to (4.3.7). With the help of equation (4.2.6), estimate (4.3.8) is now replaced by

$$\big(|q_k^+| - |q_k^-|\big) W_k^-(s_k^- - \dot{x}_\alpha) \leq \mathcal{O}(1)|\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^+|.$$

For linearly degenerate waves, (4.3.10) still holds. Together, these estimates demonstrate (4.3.5).

For a minus sign s, replace identity (4.3.6) by

$$\begin{aligned}E_k(\alpha) =& |q_k^-| W_k^-(s_k^+ - s_k^-) + \big(|q_k^+| - |q_k^-|\big) W_k^+(s_k^+ - \dot{x}_\alpha)\\ &+ |q_k^-|(W_k^+ - W_k^-)(s_k^+ - \dot{x}_\alpha).\end{aligned} \tag{4.3.15}$$

Follow the procedure outlined above for either shock, rarefaction, or linearly degenerate waves and use the interaction estimates (4.2.1), (4.2.3), and (4.2.5) of Lemma 4.2.1. The rest of the proof is identical.

The estimate of this lemma when $q_{k_\alpha}^- q_{k_\alpha}^+ < 0$ also follows the same procedure but with the estimates of Lemma 4.2.2 now replacing those of Lemma 4.2.1. ■

4.4 Isolated Waves: $E_{k_\alpha}(\alpha)$ Bounds

In this section we demonstrate on estimate for $E_{k_\alpha}(\alpha)$ when α is an isolated wave. We continue to require the total variation bound (4.3.1) and the condition (4.3.2) on the wave speeds.

Lemma 4.4.1 *For an isolated, genuinely nonlinear wave $\alpha \in \mathcal{W}\big(v(t)\big)$ satisfying $q_{k_\alpha}^- q_{k_\alpha}^+ \geq 0$, and for some sign s,*

$$\begin{aligned}E_{k_\alpha}(\alpha) \leq& -\kappa_1 d |\sigma_\alpha| |q_{k_\alpha}^s| \big(|\sigma_\alpha| + |q_{k_\alpha}^s|\big) + \mathcal{O}(1) D_\alpha(t)\\ &+ \mathcal{O}(1)\Big(|q_{k_\alpha}^s|\big(|\sigma_\alpha| + |q_{k_\alpha}^s|\big) + |\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^s|\Big).\end{aligned} \tag{4.4.1}$$

Case	wave type	relative ordering	s	σ_α	Section
1	degenerate	NA	$-$	NA	4.4.1
2	shock	$0 < q^+_{k_\alpha} < q^-_{k_\alpha}$	$+$	$-$	4.4.2
3	shock	$q^+_{k_\alpha} < q^-_{k_\alpha} < 0$	$-$	$-$	NA
4	shock	$q^+_{k_\alpha} < 0 < q^-_{k_\alpha}$	$-$	$-$	4.4.3
5	rarefaction	$0 < q^-_{k_\alpha} < q^+_{k_\alpha}$	$-$	$+$	4.4.4
6	rarefaction	$q^-_{k_\alpha} < q^+_{k_\alpha} < 0$	$+$	$+$	NA
7	rarefaction	$q^-_{k_\alpha} < 0 < q^+_{k_\alpha}$	$+$	$+$	4.4.5

Table 4.1: Case structure for Lemma 4.4.1

For an isolated, genuinely nonlinear wave satisfying $q^-_{k_\alpha} q^+_{k_\alpha} < 0$*, and some sign* s*,*

$$\begin{aligned} E_{k_\alpha}(\alpha) \leq & - \kappa_1 d |\sigma_\alpha| |q^s_{k_\alpha}| \big(|\sigma_\alpha| - |q^s_{k_\alpha}| \big) + \mathcal{O}(1) D_\alpha(t) \\ & + \mathcal{O}(1) \Big(|q^s_{k_\alpha}| \big(|\sigma_\alpha| - |q^s_{k_\alpha}| \big) + |\sigma_\alpha| \sum_{i \neq k_\alpha} |q^s_i| \Big). \end{aligned} \tag{4.4.2}$$

For an isolated, linearly degenerate wave,

$$E_{k_\alpha}(\alpha) \leq 2 D_\alpha(t) + \mathcal{O}(1) |\sigma_\alpha| \sum_{i \neq k_\alpha} |q^-_i|. \tag{4.4.3}$$

For these estimates, the wave speed $\dot{x}_\alpha$ *is only required to satisfy* (4.3.2).

We divide the proof between linearly degenerate and genuinely nonlinear waves while the latter are further subdivided into cases depending on the type of the wave and the relative ordering of the virtual waves $q^-_{k_\alpha}$, $q^+_{k_\alpha}$. This division is summarized in Table 4.4. We demonstrate the lemma for five of these cases, omitting cases 3 and 6 whose proofs are similar to cases 2 and 5.

4.4.1 Case 1: linearly degenerate

We assume that α is a wave belonging to a linearly degenerate family k_α. Refer to Figure 4.4 for a representation in phase-space of the important states. In this case, the characteristic speeds λ_{k_α} are constant along shock curves and therefore so are the Rankine-Hugoniot shock speeds. We use this observation to demonstrate the following two estimates:

$$\begin{aligned}\big|s_{k_\alpha}^- - s_{k_\alpha}(v^-,v^+)\big| =&\big|s_{k_\alpha}(w_{k_\alpha-1}^-,w_{k_\alpha}^-) - s_{k_\alpha}(v^-,v^+)\big|\\ =&\big|\lambda_{k_\alpha}(w_{k_\alpha}^-) - \lambda_{k_\alpha}(v^-)\big|\\ =&\mathcal{O}(1)|w_{k_\alpha}^- - v^-|\\ =&\mathcal{O}(1)\sum_{i\neq k_\alpha}|q_i^-|,\end{aligned}\tag{4.4.1.1}$$

and, with the help of Lemma 4.2.1,

$$\begin{aligned}|s_{k_\alpha}^+ - s_{k_\alpha}^-| =&\big|s_{k_\alpha}(w_{k_\alpha-1}^+,w_{k_\alpha}^+) - s_{k_\alpha}(w_{k_\alpha-1}^-,w_{k_\alpha}^-)\big|\\ =&\big|\lambda_{k_\alpha}(w_{k_\alpha-1}^+) - \lambda_{k_\alpha}(w_{k_\alpha-1}^-)\big|\\ =&\mathcal{O}(1)|w_{k_\alpha-1}^+ - w_{k_\alpha-1}^-|\\ =&\mathcal{O}(1)|\sigma_\alpha|\sum_{i\neq k_\alpha}|q_i^-|.\end{aligned}\tag{4.4.1.2}$$

Using the identity $W_{k_\alpha}^+ = W_{k_\alpha}^-$, we write

$$\begin{aligned}E_{k_\alpha}(\alpha) =&W_{k_\alpha}^+|q_{k_\alpha}^+|(s_{k_\alpha}^+ - \dot{x}_\alpha) - W_{k_\alpha}^-|q_{k_\alpha}^-|(s_{k_\alpha}^- - \dot{x}_\alpha)\\ =&W_{k_\alpha}^-\Big\{|q_{k_\alpha}^+|(s_{k_\alpha}^+ - \dot{x}_\alpha) - |q_{k_\alpha}^-|(s_{k_\alpha}^- - \dot{x}_\alpha)\Big\}\\ =&W_{k_\alpha}^-\Big\{|\sigma_\alpha + q_{k_\alpha}^-|\Big[\big(s_{k_\alpha}^+ - s_{k_\alpha}^-\big) + \big(s_{k_\alpha}^- - s_{k_\alpha}(v^-,v^+)\big)\\ &\qquad + \big(s_{k_\alpha}(v^-,v^+) - \dot{x}_\alpha\big)\Big]\\ &- |q_{k_\alpha}^-|\Big[\big(s_{k_\alpha}^- - s_{k_\alpha}(v^-,v^+)\big) + \big(s_{k_\alpha}(v^-,v^+) - \dot{x}_\alpha\big)\Big]\\ &+ \big(|q_{k_\alpha}^+| - |\sigma_\alpha + q_{k_\alpha}^-|\big)(s_{k_\alpha}^+ - \dot{x}_\alpha)\Big\}\\ =&W_{k_\alpha}^-\Big\{|\sigma_\alpha + q_{k_\alpha}^-|(s_{k_\alpha}^+ - s_{k_\alpha}^-) + \big(|q_{k_\alpha}^+| - |\sigma_\alpha + q_{k_\alpha}^-|\big)(s_{k_\alpha}^+ - \dot{x}_\alpha)\\ &+ \big(|\sigma_\alpha + q_{k_\alpha}^-| - |q_{k_\alpha}^-|\big)(s_{k_\alpha}^- - s_{k_\alpha}(v^-,v^+))\\ &+ \big(|\sigma_\alpha + q_{k_\alpha}^-| - |q_{k_\alpha}^-|\big)(s_{k_\alpha}(v^-,v^+) - \dot{x}_\alpha)\Big\}.\end{aligned}\tag{4.4.1.3}$$

To estimate the terms (4.4.1.3), observe $W_{k_\alpha}^- \leq 2$ by (2.2.2). According to Lemma 2.2.2, the approximations u and v remain in Ω for all t and therefore both $|\sigma_\alpha|$ and $|q_{k_\alpha}^-|$ are $\mathcal{O}(1)$ quantities. Combining these bounds with (4.4.1.2), we can estimate the first term in the braces as

$$|\sigma_\alpha + q_{k_\alpha}^-|(s_{k_\alpha}^+ - s_{k_\alpha}^-) \leq \mathcal{O}(1)|\sigma_\alpha|\sum_{i\neq k_\alpha}|q_i^-|.\tag{4.4.1.4}$$

In the second term, the quantity $(s_{k_\alpha}^+ - \dot{x}_\alpha)$ is bounded since $|\dot{x}_\alpha| \leq \Lambda$. Estimate

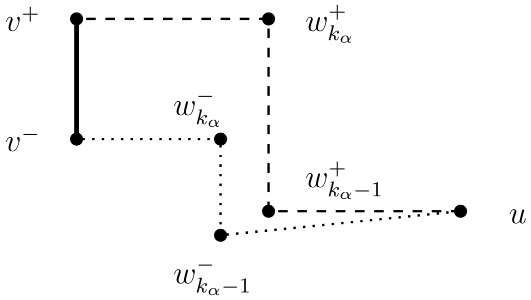

Figure 4.4: Illustration of the definitions for Case 1. The bold line represents the wave α. The dotted line represents the virtual waves q_i^- while the dashed lines represent the virtual waves q_i^+.

(4.2.5) of Lemma 4.2.1 then suffices to bound the second term.

$$\begin{aligned}\big(|q_{k_\alpha}^+| - |\sigma_\alpha + q_{k_\alpha}^-|\big)(s_{k_\alpha}^+ - \dot{x}_\alpha) \leq & \mathcal{O}(1)|q_{k_\alpha}^+ - q_{k_\alpha}^- - \sigma_\alpha| \\ = & \mathcal{O}(1)|\sigma_\alpha| \sum_{i\neq k_\alpha} |q_i^-|. \end{aligned} \tag{4.4.1.5}$$

The third term can be estimated with a combination of (4.4.1.1) and the triangle inequality:

$$\big||\sigma_\alpha + q_{k_\alpha}^-| - |q_{k_\alpha}^-|\big| \leq |\sigma_\alpha|. \tag{4.4.1.6}$$

Using the same triangle inequality, and the explicit form of the residuals in Lemma 2.3.1, after multiplying by W_k^- we can bound the last term by $2D_\alpha(t)$. Summarizing, we have shown that

$$E_{k_\alpha}(\alpha) \leq 2D_\alpha(t) + \mathcal{O}(1)|\sigma_\alpha| \sum_{i\neq k_\alpha} |q_i^-|. \tag{4.4.1.7}$$

This completes the proof of the first case.

4.4.2 Case 2: shock with $0 < q_{k_\alpha}^+ < q_{k_\alpha}^-$

Assume that α is a shock wave with $0 < q_{k_\alpha}^+ < q_{k_\alpha}^-$. We note that since the wave is isolated, v^- and v^+ equal the states to the left and right of the wave α. As a preliminary for the first estimate we introduce the states and shock speeds

$$\begin{aligned} \widetilde{w} &= S_{k_\alpha}(q_{k_\alpha}^+ - \sigma_\alpha)(w_{k_\alpha-1}^+), & \widetilde{s} &= s_{k_\alpha}(w_{k_\alpha-1}^+, \widetilde{w}), \\ w^* &= S_{k_\alpha}(-\sigma_\alpha)(w_{k_\alpha}^+), & s^* &= s_{k_\alpha}(w_{k_\alpha}^+, w^*). \end{aligned} \tag{4.4.2.1}$$

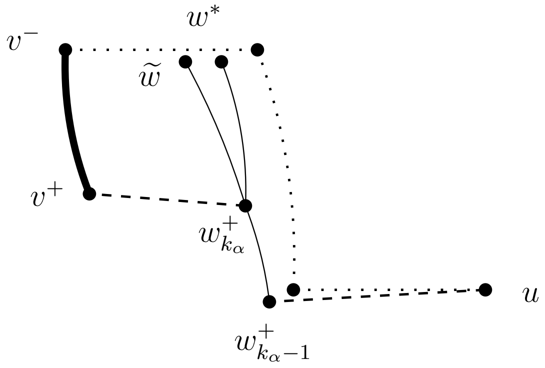

Figure 4.5: Illustration of the definitions (4.4.2.1) for Case 2. The solid line represents the wave curves joining the states in definitions (4.4.2.1).

Figure 4.5 illustrates these definitions in phase-space. The estimates we obtain involve q_i^+, i.e. the sign s is positive. The wave speeds $s^-_{k_\alpha} = s_{k_\alpha}(w^-_{k_\alpha-1}, w^-_{k_\alpha})$ and $\widetilde{s} = s_{k_\alpha}(w^+_{k_\alpha-1}, \widetilde{w})$ differ by the distance between $w^-_{k_\alpha-1}$ and $w^+_{k_\alpha-1}$ and by the length of the shock curves connecting these states to $w^-_{k_\alpha}$ and $\widetilde{w}$ respectively. This observation and the estimates of Lemma 4.2.1 demonstrate

$$\begin{aligned} |s^-_{k_\alpha} - \widetilde{s}| &= \mathcal{O}(1)\Big(|w^+_{k_\alpha-1} - w^-_{k_\alpha-1}| + |q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha|\Big) \\ &\leq \mathcal{O}(1)\Big(|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big) + |\sigma_\alpha| \sum_{i\neq k_\alpha} |q_i^+|\Big). \end{aligned} \tag{4.4.2.2}$$

The shock speeds $s^* = s_{k_\alpha}(w^+_{k_\alpha}, w^*)$ and $s_{k_\alpha}(v^-, v^+)$ are both associated with states separated by shock curves of length $|\sigma_\alpha|$. We can therefore use the distance between the states $w^+_{k_\alpha}$ and v^+ to bound the difference

$$\big|s^* - s_{k_\alpha}(v^+, v^-)\big| = \mathcal{O}(1)|w^+_{k_\alpha} - v^+| = \mathcal{O}(1) \sum_{i\neq k_\alpha} |q_i^+|. \tag{4.4.2.3}$$

Recall inequality (1.1.26) involving the positive constant d. If $q = q^+_{k_\alpha}$, $p = \sigma_\alpha$, $u = w^+_{k_\alpha-1}$, then that inequality becomes

$$d|q^+_{k_\alpha}| < s^* - \widetilde{s}. \tag{4.4.2.4}$$

Making use of the identity $W^+_{k_\alpha} - W^-_{k_\alpha} = \kappa_1|\sigma_\alpha|$ from Lemma 2.2.1 and the fact that

$q^-_{k_\alpha}$, $q^+_{k_\alpha}$ are positive and σ_α is negative, we expand (4.1.5)

$$\begin{aligned}
E_{k_\alpha}(\alpha) &= W^+_{k_\alpha}|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - W^-_{k_\alpha}|q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha) \\
&= W^+_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\} \\
&\qquad + \kappa_1|\sigma_\alpha||q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha) \\
&= W^+_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\} \\
&\qquad + \kappa_1|\sigma_\alpha|\big(|q^-_{k_\alpha}| - |\sigma_\alpha| - |q^+_{k_\alpha}|\big)(s^-_{k_\alpha} - \dot{x}_\alpha) \\
&\qquad + \kappa_1|\sigma_\alpha|\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big)(s^-_{k_\alpha} - \dot{x}_\alpha) \\
&\le W^+_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\} \\
&\qquad + \kappa_1|\sigma_\alpha|\big|q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha\big||s^-_{k_\alpha} - \dot{x}_\alpha| \\
&\qquad + \kappa_1|\sigma_\alpha|\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big)\Big(\big(s^-_{k_\alpha} - \widetilde{s}\big) + \big(\widetilde{s} - s^*\big) \\
&\qquad + \big(s^* - s_{k_\alpha}(v^-, v^+)\big) + \big(s_{k_\alpha}(v^-, v^+) - \dot{x}_\alpha\big)\Big). \qquad (4.4.2.5)
\end{aligned}$$

There are three terms in the final expression for $E_{k_\alpha}(\alpha)$. On the second term, we use the bound $\kappa_1|\sigma_\alpha| \le 1$ from Lemma 4.2.4 and equation (4.2.2) of Lemma 4.2.1. Together these show that

$$\begin{aligned}
&\kappa_1|\sigma_\alpha|\big|q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha\big||s^-_{k_\alpha} - \dot{x}_\alpha| \\
&\qquad \le \mathcal{O}(1)\Big(|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| + |q^+_{k_\alpha}\big) + |\sigma_\alpha|\sum_{i \ne k_\alpha}|q^+_i|\Big). \qquad (4.4.2.6)
\end{aligned}$$

Notice that Lemma 4.2.4 provides us with the upper bound $\kappa_1\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big) \le 2$. Use identities (4.4.2.2), (4.4.2.3), and (4.4.2.4) on the third term of (4.4.2.5) to yield

$$\begin{aligned}
&\kappa_1|\sigma_\alpha|\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big) \times \\
&\quad \Big(\big(s^-_{k_\alpha} - \widetilde{s}\big) + \big(\widetilde{s} - s^*\big) + \big(s^* - s_{k_\alpha}(v^-, v^+)\big) + \big(s_{k_\alpha}(v^-, v^+) - \dot{x}_\alpha\big)\Big) \\
&\qquad \le \mathcal{O}(1)\Big(|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| + |q^+_{k_\alpha}\big) + |\sigma_\alpha|\sum_{i \ne k_\alpha}|q^+_i|\Big) \qquad (4.4.2.7) \\
&\qquad\quad - \kappa_1 d|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big) + 2|\sigma_\alpha|\big|s_{k_\alpha}(v^-, v^+) - \dot{x}_\alpha\big|.
\end{aligned}$$

According to Lemma 2.3.1

$$|\sigma_\alpha|\big|s_{k_\alpha}(v^-, v^+) - \dot{x}_\alpha\big| = D_\alpha(t). \qquad (4.4.2.8)$$

Using this fact in (4.4.2.7), and combining the result with (4.4.2.6), we obtain

$$\begin{aligned}E_{k_\alpha}(\alpha) \le & W^+_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha)-|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\ & +\mathcal{O}(1)\Big(|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha|+|q^+_{k_\alpha}|\big)+|\sigma_\alpha|\sum_{i\neq k_\alpha}|q^+_i|\Big)\\ & -\kappa_1 d|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha|+|q^+_{k_\alpha}|\big)+2D_\alpha(t). \end{aligned} \tag{4.4.2.9}$$

From our knowledge of the signs of $q^+_{k_\alpha}, q^-_{k_\alpha}$, and σ_α, we expand the first term in expression (4.4.2.9):

$$\begin{aligned} & W^+_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha)-|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\ & \quad = W^+_{k_\alpha}\Bigg\{q^+_{k_\alpha}\Big[\big(s^+_{k_\alpha}-s^*\big)+\big(s^*-s_{k_\alpha}(v^-,v^+)\big)+\big(s_{k_\alpha}(v^-,v^+)-\dot{x}_\alpha\big)\Big]\\ & \qquad -(q^+_{k_\alpha}-\sigma_\alpha)\Big[\big(s^-_{k_\alpha}-\widetilde{s}\big)+\big(\widetilde{s}-s^*\big)\\ & \qquad +\big(s^*-s_{k_\alpha}(v^-,v^+)\big)+\big(s_{k_\alpha}(v^-,v^+)-\dot{x}_\alpha\big)\Big]\\ & \qquad -(q^-_{k_\alpha}-q^+_{k_\alpha}+\sigma_\alpha)(s^-_{k_\alpha}-\dot{x}_\alpha)\Bigg\}\\ & \quad = W^+_{k_\alpha}\Big\{q^+_{k_\alpha}(s^+_{k_\alpha}-s^*)-(q^+_{k_\alpha}-\sigma_\alpha)(\widetilde{s}-s^*)\Big\}\\ & \qquad +W^+_{k_\alpha}\Big\{-(q^+_{k_\alpha}-\sigma_\alpha)(s^-_{k_\alpha}-\widetilde{s})\\ & \qquad\quad +\sigma_\alpha\Big[\big(s^*-s_{k_\alpha}(v^-,v^+)\big)+\big(s_{k_\alpha}(v^-,v^+)-\dot{x}_\alpha\big)\Big]\\ & \qquad\quad -(q^-_{k_\alpha}-q^+_{k_\alpha}+\sigma_\alpha)(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}. \end{aligned} \tag{4.4.2.10}$$

Lemma 4.2.5, with $\overline{w}=w^+_{k_\alpha-1}$, $\sigma=q^+_{k_\alpha}$, and $\sigma'=-\sigma_\alpha$, followed by the bound (2.2.2), $W^+_{k_\alpha}\le 2$, imply a bound on the first term in (4.4.2.10):

$$|W^+_{k_\alpha}|\Big|q^+_{k_\alpha}(s^+_{k_\alpha}-s^*)-(q^+_{k_\alpha}-\sigma_\alpha)(\widetilde{s}-s^*)\Big|\le\mathcal{O}(1)|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha|+|q^+_{k_\alpha}|\big). \tag{4.4.2.11}$$

Applying Lemma 4.2.1, the bounds (4.4.2.2), (4.4.2.3), (4.4.2.11), (4.2.2), and our characterization (4.4.2.8) of the discrepancy, we can construct an upper bound for (4.4.2.10),

$$\begin{aligned} & W^+_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha)-|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\ & \qquad \le\mathcal{O}(1)\Big(|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha|+|q^+_{k_\alpha}|\big)+|\sigma_\alpha|\sum_{i\neq k_\alpha}|q^+_i|\Big)+2D_\alpha(t). \end{aligned} \tag{4.4.2.12}$$

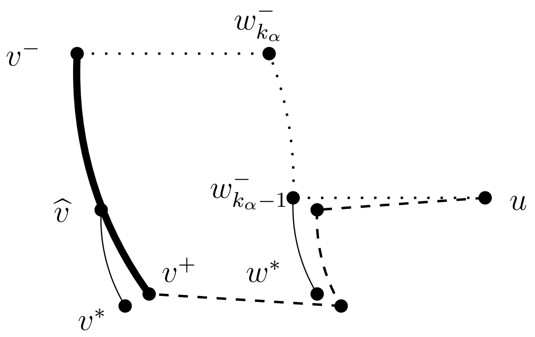

Figure 4.6: Illustration of the definitions (4.4.3.1) for case 4.

Estimates (4.4.2.9) and (4.4.2.12) give

$$\begin{aligned} E_{k_\alpha}(\alpha) \leq & -\kappa_1 d|\sigma_\alpha||q_{k_\alpha}^+|\big(|\sigma_\alpha| + |q_{k_\alpha}^+|\big) + 4D_\alpha(t) \\ & + \mathcal{O}(1)\Big(|\sigma_\alpha||q_{k_\alpha}^+|\big(|\sigma_\alpha| + |q_{k_\alpha}^+|\big) + |\sigma_\alpha| \sum_{i\neq k_\alpha} |q_i^+|\Big). \end{aligned} \tag{4.4.2.13}$$

4.4.3 Case 4: shock with $q_{k_\alpha}^+ < 0 < q_{k_\alpha}^-$

Assume that α is a shock wave for which $q_{k_\alpha}^+ < 0 < q_{k_\alpha}^-$. The desired estimate will be obtained with a negative sign s.

We introduce the following states obtained from shock curves of the k_α-th family

$$\begin{aligned} w^* &= S_{k_\alpha}(q_{k_\alpha}^- + \sigma_\alpha)(w_{k_\alpha-1}^-), & s^* &= s_{k_\alpha}(w_{k_\alpha-1}^-, w^*), \\ \widetilde{w} &= S_{k_\alpha}(\sigma_\alpha)(w_{k_\alpha}^-), & \widetilde{s} &= s_{k_\alpha}(w_{k_\alpha}^-, \widetilde{w}), \\ \widehat{v} &= S_{k_\alpha}(-q_{k_\alpha}^-)(v^-), & \widehat{\eta} &= s_{k_\alpha}(v^-, \widehat{v}), \\ v^* &= S_{k_\alpha}(q_{k_\alpha}^- + \sigma_\alpha)(\widehat{v}), & \eta^* &= s_{k_\alpha}(\widehat{v}, v^*). \end{aligned} \tag{4.4.3.1}$$

For the pairs of shock speeds used below, the states defining them are separated by the same distance along shock curves. Using this observation, we have

$$\begin{aligned} \big|\widetilde{s} - s_{k_\alpha}(v^-, v^+)\big| &= \big|s_{k_\alpha}(w_{k_\alpha}^-, \widetilde{w}) - s_{k_\alpha}(v^-, v^+)\big| \\ &= \mathcal{O}(1)|w_{k_\alpha}^- - v^-| \\ &= \mathcal{O}(1) \sum_{i\neq k_\alpha} |q_i^-|, \end{aligned} \tag{4.4.3.2}$$

$$\begin{aligned}|s_{k_\alpha}^- - \widehat{\eta}| &= \big|s_{k_\alpha}(w_{k_\alpha-1}^-, w_{k_\alpha}^-) - s_{k_\alpha}(v^-, \widehat{v})\big| \\ &= \mathcal{O}(1)|w_{k_\alpha}^- - v^-| \\ &= \mathcal{O}(1)\sum_{i\neq k_\alpha}|q_i^-|, \end{aligned}\tag{4.4.3.3}$$

and

$$\begin{aligned}|s^* - \eta^*| &= \big|s_{k_\alpha}(w_{k_\alpha-1}^-, w^*) - s_{k_\alpha}(\widehat{v}, v^*)\big| \\ &= \mathcal{O}(1)|w_{k_\alpha-1}^- - \widehat{v}| \\ &= \mathcal{O}(1)|w_{k_\alpha}^- - v^-| \\ &= \mathcal{O}(1)\sum_{i\neq k_\alpha}|q_i^-|. \end{aligned}\tag{4.4.3.4}$$

The same observation combined with the bounds of Lemma 4.2.2 leads to

$$\begin{aligned}|s_{k_\alpha}^+ - s^*| &= \mathcal{O}(1)\Big(|w_{k_\alpha-1}^+ - w_{k_\alpha-1}^-| + |q_{k_\alpha}^+ - q_{k_\alpha}^- - \sigma_\alpha|\Big) \\ &= \mathcal{O}(1)\Big(|\sigma_\alpha||q_{k_\alpha}^-|\big(|\sigma_\alpha| - |q_{k_\alpha}^-|\big) + |\sigma_\alpha|\sum_{i\neq k_\alpha}|q_i^-|\Big). \end{aligned}\tag{4.4.3.5}$$

Consider property (1.1.27) of the constant d when $q = q_{k_\alpha}^-$, $p = -\sigma_\alpha - q_{k_\alpha}^-$, and $u = w_{k_\alpha}^-$. We find

$$d|\sigma_\alpha + q_{k_\alpha}^-| = d\big(|\sigma_\alpha| - |q_{k_\alpha}^-|\big) \leq s_{k_\alpha}^- - \widetilde{s}. \tag{4.4.3.6}$$

The approximations $u(t)$ are in L^1 for all t and therefore, being piecewise constant, the approximations must vanish outside some compact set. In the k_α-th coordinate, the sum of the increasing variation is then equal to the sum of the decreasing variation. In particular, for the wave α, we have

$$2|\sigma_\alpha| \leq \sum_{\substack{\beta\in\mathcal{W}(v)\\ k_\beta = k_\alpha}}|\sigma_\beta|. \tag{4.4.3.7}$$

Using this bound with (2.2.8), we find that

$$|\sigma_\alpha| \leq B_{k_\alpha}^- + B_{k_\alpha}^+. \tag{4.4.3.8}$$

We present a preliminary decomposition for $E_{k_\alpha}(\alpha)$ into three terms. Note that $A_{k_\alpha}^+ = A_{k_\alpha}^-$ since the terms in (2.2.4) involve only waves from families other than k_α.

Suppressing the time dependence, we have

$$\begin{aligned}
E_{k_\alpha}(\alpha) =& W^+_{k_\alpha}|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - W^-_{k_\alpha}|q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha) \\
=& \Big(1 + \kappa_1\big(A^+_{k_\alpha} + B^+_{k_\alpha}\big) + \kappa_2\big(Q(u) + Q(v)\big)\Big)|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) \\
&- \Big(1 + \kappa_1\big(A^-_{k_\alpha} + B^-_{k_\alpha}\big) + \kappa_2\big(Q(u) + Q(v)\big)\Big)|q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha) \\
=& \Big(1 + \kappa_1 A^-_{k_\alpha} + \kappa_2\big(Q(u) + Q(v)\big)\Big)\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\} \\
&- \kappa_1|q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big(B^-_{k_\alpha} + B^+_{k_\alpha} - B^+_{k_\alpha}\Big) \\
&+ \kappa_1|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha)B^+_{k_\alpha} \\
=& \Big(1 + \kappa_1 A^-_{k_\alpha} + \kappa_2\big(Q(u) + Q(v)\big)\Big)\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\} \\
&- \kappa_1|q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big(B^-_{k_\alpha} + B^+_{k_\alpha}\Big) \\
&+ \kappa_1 B^+_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) + |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\}. \qquad (4.4.3.9)
\end{aligned}$$

We begin by remarking that the bound $W^+_{k_\alpha} \leq 2$, from equation (2.2.2), implies that

$$\Big(1 + \kappa_1 A^-_{k_\alpha} + \kappa_2\big(Q(u) + Q(v)\big)\Big) \leq 2. \qquad (4.4.3.10)$$

For the other half of the first term in (4.4.3.9), using $q^+_{k_\alpha} \leq 0 \leq q^-_{k_\alpha}$, and $\sigma_\alpha < 0$, we have

$$\begin{aligned}
&|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha) \\
&\quad = -(q^-_{k_\alpha} + \sigma_\alpha)\big((s^+_{k_\alpha} - s^*) + (s^* - \eta^*) + (\eta^* - \dot{x}_\alpha)\big) \\
&\qquad - q^-_{k_\alpha}\big((s^-_{k_\alpha} - \widehat{\eta}) + (\widehat{\eta} - \dot{x}_\alpha)\big) + (-q^+_{k_\alpha} + q^-_{k_\alpha} + \sigma_\alpha)(s^+_{k_\alpha} - \dot{x}_\alpha) \\
&\quad = -(q^-_{k_\alpha} + \sigma_\alpha)(\eta^* - \dot{x}_\alpha) - q^-_{k_\alpha}(\widehat{\eta} - \dot{x}_\alpha) \\
&\qquad - (q^-_{k_\alpha} + \sigma_\alpha)\big((s^+_{k_\alpha} - s^*) + (s^* - \eta^*)\big) \\
&\qquad - q^-_{k_\alpha}(s^-_{k_\alpha} - \widehat{\eta}) + (-q^+_{k_\alpha} + q^-_{k_\alpha} + \sigma_\alpha)(s^+_{k_\alpha} - \dot{x}_\alpha). \qquad (4.4.3.11)
\end{aligned}$$

We recognize that the first two terms in the final equality in (4.4.3.11) form the quadratic shock remainder $Q_{\text{shk}}(q^-_{k_\alpha}, \dot{x}_\alpha)$. In Lemma 4.2.6, we showed that the supremum of the shock remainder over $q^-_{k_\alpha} \in [0, |\sigma_\alpha|]$ is precisely $D_\alpha(t)$. Notice that $|q^-_{k_\alpha}|, |q^-_{k_\alpha} + \sigma_\alpha|$ are both bounded above by $|\sigma_\alpha|$. Using this fact and identities (4.4.3.3), (4.4.3.4), and (4.4.3.5) we can produce upper bounds on the third and fourth terms. The last term in (4.4.3.11) can be bounded by the estimate (4.2.1) of Lemma 4.2.2. In this manner, we have demonstrated that the first term in the last equality of (4.4.3.9) is bounded by

$$\begin{aligned}
&\Big(1+\kappa_1 A^-_{k_\alpha} + \kappa_2\big(Q(u)+Q(v)\big)\Big)\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\
&\qquad \le 2D_\alpha(t) + \mathcal{O}(1)\Big(|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha| - |q^-_{k_\alpha}|\big) + |\sigma_\alpha|\sum_{i\ne k_\alpha}|q^-_i|\Big). \qquad (4.4.3.12)
\end{aligned}$$

We now turn to the second term in the last equality of (4.4.3.9) and rewrite it as

$$\begin{aligned}
&-\kappa_1|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\big(B^-_{k_\alpha}+B^+_{k_\alpha}\big)\\
&\quad = -\kappa_1|q^-_{k_\alpha}|(s^-_{k_\alpha}-\widetilde{s})\big(B^-_{k_\alpha}+B^+_{k_\alpha}\big)\\
&\quad - \kappa_1|q^-_{k_\alpha}|\Big(\big(\widetilde{s}-s_{k_\alpha}(v^+,v^-)\big) + \big(s_{k_\alpha}(v^+,v^-)-\dot{x}_\alpha\big)\Big)\big(B^-_{k_\alpha}+B^+_{k_\alpha}\big). \qquad (4.4.3.13)
\end{aligned}$$

We note the inequalities $\kappa_1(B^-_{k_\alpha}+B^+_{k_\alpha}) \le 2$ and $|q^-_{k_\alpha}| \le |\sigma_\alpha|$. Using these in conjunction with estimates (2.3.11), (4.4.3.2), (4.4.3.6), and (4.4.3.8), (4.4.3.13) can be estimated as

$$\begin{aligned}
&-\kappa_1|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big(B^-_{k_\alpha}+B^+_{k_\alpha}\Big)\\
&\qquad \le -\kappa_1 d|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha|-|q^-_{k_\alpha}|\big) + 2D_\alpha(t) + \mathcal{O}(1)|\sigma_\alpha|\sum_{i\ne k_\alpha}|q^-_i|. \qquad (4.4.3.14)
\end{aligned}$$

We expand the remaining term in (4.4.3.9) using our assumption that $q^-_{k_\alpha}, -q^+_{k_\alpha}$, and $-\sigma_\alpha$ are positive:

$$\begin{aligned}
&\kappa_1 B^+_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha) + |q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\
&\quad =\kappa_1 B^+_{k_\alpha}\Big\{-(q^-_{k_\alpha}+\sigma_\alpha)\Big[\big(s^+_{k_\alpha}-s^*\big) + \big(s^*-\widetilde{s}\big)\\
&\qquad\qquad + \big(\widetilde{s}-s_{k_\alpha}(v^+,v^-)\big) + \big(s_{k_\alpha}(v^+,v^-)-\dot{x}_\alpha\big)\Big]\\
&\qquad + q^-_{k_\alpha}\Big[\big(s^-_{k_\alpha}-\widetilde{s}\big) + \big(\widetilde{s}-s_{k_\alpha}(v^+,v^-)\big) + \big(s_{k_\alpha}(v^+,v^-)-\dot{x}_\alpha\big)\Big]\\
&\qquad - (q^+_{k_\alpha}-q^-_{k_\alpha}-\sigma_\alpha)(s^+_{k_\alpha}-\dot{x}_\alpha)\Big\}\\
&\quad =\kappa_1 B^+_{k_\alpha}\Big\{-(q^-_{k_\alpha}+\sigma_\alpha)(s^*-\widetilde{s}) + q^-_{k_\alpha}(s^-_{k_\alpha}-\widetilde{s})\\
&\qquad\qquad - (q^-_{k_\alpha}+\sigma_\alpha)(s^+_{k_\alpha}-s^*) - (q^+_{k_\alpha}-q^-_{k_\alpha}-\sigma_\alpha)(s^+_{k_\alpha}-\dot{x}_\alpha)\\
&\qquad\qquad - \sigma_\alpha\Big[\big(\widetilde{s}-s_{k_\alpha}(v^+,v^-)\big) + \big(s_{k_\alpha}(v^+,v^-)-\dot{x}_\alpha\big)\Big]\Big\}. \qquad (4.4.3.15)
\end{aligned}$$

According to Lemma 2.2.2, the coefficient $\kappa_1 B^+_{k_\alpha}$ multiplying this expression is bounded by 2. Ignoring this coefficient, the first two terms are precisely expression (4.2.19)

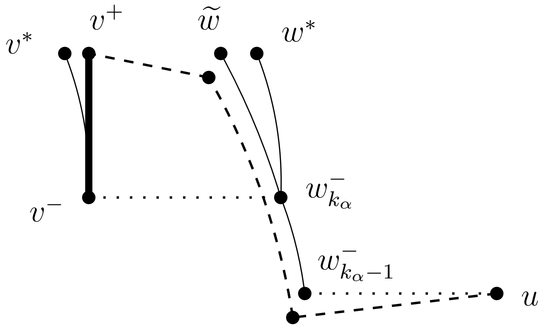

Figure 4.7: Illustration of definitions (4.4.4.1) for the fifth case.

from Lemma 4.2.5 with $\overline{w} = w^-_{k_\alpha-1}$, $\sigma = q^-_{k_\alpha}$, and $\sigma' = \sigma_\alpha$. This shows that

$$\big| -(q^-_{k_\alpha} + \sigma_\alpha)(s^* - \widetilde{s}) + q^-_{k_\alpha}(s^-_{k_\alpha} - \widetilde{s})\big| = \mathcal{O}(1)|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha| - |q^-_{k_\alpha}|\big). \tag{4.4.3.16}$$

The next three terms in (4.4.3.15) are controlled by (4.4.3.2), (4.4.3.5), and Lemma 4.2.2. The remaining term is the discrepancy of the wave α. Together, these demonstrate that

$$\begin{aligned}&\kappa_1 B^+_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) + |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\}\\ &\qquad\le 2D_\alpha(t) + \mathcal{O}(1)\Big(|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha| - |q^-_{k_\alpha}|\big) + |\sigma_\alpha|\sum_{i\neq k_\alpha}|q^-_i|\Big).\end{aligned} \tag{4.4.3.17}$$

The final estimate for $E_{k_\alpha}(\alpha)$ is the sum of the estimates (4.4.3.12), (4.4.3.14), and (4.4.3.17) for the three terms in (4.4.3.9). We have shown that, in case 4, the estimate is

$$\begin{aligned}E_{k_\alpha}(\alpha) \le &- \kappa_1 d|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha| - |q^-_{k_\alpha}|\big) + 6D_\alpha(t)\\ &+ \mathcal{O}(1)\Big(|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha| - |q^-_{k_\alpha}|\big) + |\sigma_\alpha|\sum_{i\neq k_\alpha}|q^-_i|\Big).\end{aligned} \tag{4.4.3.18}$$

4.4.4 Case 5: rarefaction with $0 < q^-_{k_\alpha} < q^+_{k_\alpha}$

Suppose that α is a pure rarefaction wave with $0 < q^-_{k_\alpha} < q^+_{k_\alpha}$. We define the intermediate states and speeds

$$\begin{aligned}\widetilde{w} &= S_{k_\alpha}(q^-_{k_\alpha} + \sigma_\alpha)(w^-_{k_\alpha-1}), & \widetilde{s} &= s_{k_\alpha}(w^-_{k_\alpha-1}, \widetilde{w}),\\ w^* &= S_{k_\alpha}(\sigma_\alpha)(w^-_{k_\alpha}), & s^* &= s_{k_\alpha}(w^-_{k_\alpha}, w^*),\end{aligned} \tag{4.4.4.1}$$

and

$$v^* = S_{k_\alpha}(\sigma_\alpha)(v^-). \tag{4.4.4.2}$$

These states are illustrated in Figure 4.7. From the definition of w^* and v^*, we have

$$\begin{aligned}\big|s^* - s_{k_\alpha}(v^-, v^*)\big| &= \big|s_{k_\alpha}(w^-_{k_\alpha}, w^*) - s_{k_\alpha}(v^-, v^*)\big| \\ &= \mathcal{O}(1)|w^-_{k_\alpha} - v^-| \\ &= \mathcal{O}(1)\sum_{i\neq k_\alpha} |q_i^-|. \end{aligned} \tag{4.4.4.3}$$

The same method is now used in conjunction with the estimates (4.2.3) of Lemma 4.2.1 to yield

$$\begin{aligned}|s^+_{k_\alpha} - \widetilde{s}| &= \big|s_{k_\alpha}(w^+_{k_\alpha-1}, w^+_{k_\alpha}) - s_{k_\alpha}(w^-_{k_\alpha-1}, \widetilde{w})\big| \\ &= \mathcal{O}(1)\Big(|w^+_{k_\alpha-1} - w^-_{k_\alpha-1}| + |q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha|\Big) \\ &= \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha| + |q^-_{k_\alpha}|\big) + |\sigma_\alpha|\sum_{i\neq k_\alpha}|q_i^-|\Big). \end{aligned} \tag{4.4.4.4}$$

Property (1.1.26) of the constant d shows that for $q = q^-_{k_\alpha}$, $p = \sigma_\alpha$, and $u = w^-_{k_\alpha-1}$

$$d|q^-_{k_\alpha}| < s^* - \widetilde{s}. \tag{4.4.4.5}$$

Using the fact that, to second-order, the shock speed is the average of the characteristic speeds of the states on the left and right hand sides, we have

$$\left|s_{k_\alpha}(v^-, v^*) - \frac{1}{2}\big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(v^*)\big)\right| = \mathcal{O}(1)|\sigma_\alpha|^2. \tag{4.4.4.6}$$

For $v^+ = R_{k_\alpha}(\sigma_\alpha)(v^-)$ and $v^* = S_{k_\alpha}(\sigma_\alpha)(v^-)$, our choice of parameterization (1.1.15) implies that

$$\lambda_{k_\alpha}(v^+) - \lambda_{k_\alpha}(v^-) = \sigma_\alpha = \lambda_{k_\alpha}(v^*) - \lambda_{k_\alpha}(v^-). \tag{4.4.4.7}$$

In particular,

$$\lambda_{k_\alpha}(v^+) = \lambda_{k_\alpha}(v^*). \tag{4.4.4.8}$$

Lemma 2.2.1 states that under the assumptions of Case 5, $W^+_{k_\alpha} - W^-_{k_\alpha} = \kappa_1|\sigma_\alpha|$. Using this fact, identity (4.4.4.8), and the positivity of $q^+_{k_\alpha}$, $q^-_{k_\alpha}$, and σ_α, we expand the local contribution to the derivative of Φ as

$$\begin{aligned}
E_{k_\alpha}(\alpha) &= W^+_{k_\alpha}|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha) - W^-_{k_\alpha}|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\\
&= \kappa_1|\sigma_\alpha||q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha) + W^-_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\
&\le \kappa_1|\sigma_\alpha|(q^-_{k_\alpha}+\sigma_\alpha)\Big[\big(s^+_{k_\alpha}-\widetilde{s}\big) + \big(\widetilde{s}-s^*\big) + \big(s^* - s_{k_\alpha}(v^-,v^*)\big)\\
&\qquad\qquad + \Big(s_{k_\alpha}(v^-,v^*) - \frac{1}{2}\big(\lambda_{k_\alpha}(v^-)+\lambda_{k_\alpha}(v^*)\big)\Big)\\
&\qquad\qquad + \Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^-)+\lambda_{k_\alpha}(v^+)\big) - \dot{x}_\alpha\Big)\Big]\\
&\quad + \kappa_1|\sigma_\alpha||q^+_{k_\alpha}-q^-_{k_\alpha}-\sigma_\alpha||s^+_{k_\alpha}-\dot{x}_\alpha|\\
&\quad + W^-_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}. \qquad (4.4.4.9)
\end{aligned}$$

In the first term, use identity (4.4.4.5) to obtain

$$\kappa_1|\sigma_\alpha|(q^-_{k_\alpha}+\sigma_\alpha)(\widetilde{s}-s^*) \le -\kappa_1 d|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha|+|q^-_{k_\alpha}|\big). \qquad (4.4.4.10)$$

According to Lemma 4.2.4, $\kappa_1(q^-_{k_\alpha}+\sigma_\alpha)\le 2$. Using this bound and (4.4.4.3), (4.4.4.4), and (4.4.4.6) we can bound three other components of the first term (4.4.4.9). From Lemma 2.3.2, we recognize the lower bound on the discrepancy as the last component of the last term in equation (4.4.4.9). Finally, we can use the estimate $\kappa_1|\sigma_\alpha|\le 1$ from Lemma 4.2.4 and the bound on $|q^+_{k_\alpha}-q^-_{k_\alpha}-\sigma_\alpha|$ from Lemma 4.2.1 to control the second term in (4.4.4.9). Together, these remarks show that

$$\begin{aligned}
E_{k_\alpha}(\alpha) \le &- \kappa_1 d|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha|+|q^-_{k_\alpha}|\big) + 2D_\alpha(t)\\
&+ \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha|+|q^-_{k_\alpha}|\big) + |\sigma_\alpha|\sum_{i\neq k_\alpha}|q^-_i|\Big)\\
&+ W^-_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}. \qquad (4.4.4.11)
\end{aligned}$$

To study the last term in (4.4.4.11), use the positivity of $q^-_{k_\alpha}$, $q^+_{k_\alpha}$, and σ_α to rewrite it as

$$\begin{aligned}
&W^-_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\
&\quad = W^-_{k_\alpha}\Big\{(q^-_{k_\alpha}+\sigma_\alpha)\Big[\big(s^+_{k_\alpha}-\widetilde{s}\big) + \big(\widetilde{s}-s^*\big) + \big(s^*-s_{k_\alpha}(v^-,v^*)\big) + \big(s_{k_\alpha}(v^-,v^*)-\dot{x}_\alpha\big)\Big]\\
&\qquad - q^-_{k_\alpha}\Big[\big(s^-_{k_\alpha}-s^*\big) + \big(s^*-s_{k_\alpha}(v^-,v^*)\big) + \big(s_{k_\alpha}(v^-,v^*)-\dot{x}_\alpha\big)\Big]\\
&\qquad + (q^+_{k_\alpha}-q^-_{k_\alpha}-\sigma_\alpha)(s^+_{k_\alpha}-\dot{x}_\alpha)\Big\}
\end{aligned}$$

$$
\begin{aligned}
=&W_{k_\alpha}^-\Big\{(q_{k_\alpha}^- + \sigma_\alpha)(\widetilde{s} - s^*) - q_{k_\alpha}^-(s_{k_\alpha}^- - s^*)\Big\}\\
&+ W_{k_\alpha}^-\bigg\{(q_{k_\alpha}^- + \sigma_\alpha)(s_{k_\alpha}^+ - \widetilde{s}) + (q_{k_\alpha}^+ - q_{k_\alpha}^- - \sigma_\alpha)(s_{k_\alpha}^+ - \dot{x}_\alpha)\\
&\qquad + \sigma_\alpha\big[\big(s^* - s_{k_\alpha}(v^-, v^*)\big) + \big(s_{k_\alpha}(v^-, v^*) - \dot{x}_\alpha\big)\big]\bigg\}. \qquad (4.4.4.12)
\end{aligned}
$$

Lemma 4.2.5, with $\overline{w} = w_{k_\alpha - 1}^-$, $\sigma = q_{k_\alpha}^-$, and $\sigma' = \sigma_\alpha$ gives

$$
\big|(q_{k_\alpha}^- + \sigma_\alpha)(\widetilde{s} - s^*) - q_{k_\alpha}^-(s_{k_\alpha}^- - s^*)\big| = \mathcal{O}(1)|\sigma_\alpha||q_{k_\alpha}^-|\big(|\sigma_\alpha| + |q_{k_\alpha}^-|\big), \qquad (4.4.4.13)
$$

which is precisely the first term of (4.4.4.12), up the constant $W_{k_\alpha}^- \leq 2$. Ignoring the multiplier $W_{k_\alpha}^-$, we can control the third term with the help of identity (4.4.4.4). The fourth term, in combination with (4.4.4.3), is of the order of $|\sigma_\alpha| \sum |q_i^-|$. The last term in (4.4.4.12) can clearly be bounded by the interaction estimates of Lemma 4.2.1. For the term involving the wave speed $\dot{x}_\alpha$, use equations (4.4.4.6), (4.4.4.8), and Lemma 2.3.2 to bound it as

$$
\begin{aligned}
\sigma_\alpha(s_{k_\alpha}(v^-, v^*) - \dot{x}_\alpha) = \sigma_\alpha\bigg[&\Big(s_{k_\alpha}(v^-, v^*) - \frac{1}{2}\big(\lambda_{k_\alpha}(v^-) - \lambda_{k_\alpha}(v^*)\big)\Big)\\
&+ \Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^-) - \lambda_{k_\alpha}(v^+)\big) - \dot{x}_\alpha\Big)\bigg]\\
\leq{}& \mathcal{O}(1)|\sigma_\alpha|^3 + D_\alpha(t). \qquad (4.4.4.14)
\end{aligned}
$$

These observations on the terms in (4.4.4.12) demonstrate that

$$
\begin{aligned}
&W_{k_\alpha}^-\Big\{|q_{k_\alpha}^+|(s_{k_\alpha}^+ - \dot{x}_\alpha) - |q_{k_\alpha}^-|(s_{k_\alpha}^- - \dot{x}_\alpha)\Big\}\\
&\quad\leq 2D_\alpha(t) + \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q_{k_\alpha}^-|\big(|\sigma_\alpha| + |q_{k_\alpha}^-|\big) + |\sigma_\alpha|\sum_{i \neq k_\alpha}|q_i^-|\Big). \qquad (4.4.4.15)
\end{aligned}
$$

Combining this with our earlier estimate, (4.4.4.11), leads us to

$$
\begin{aligned}
E_{k_\alpha}(\alpha) \leq& -\kappa_1 d|\sigma_\alpha||q_{k_\alpha}^-|\big(|\sigma_\alpha| + |q_{k_\alpha}^-|\big) + 4D_\alpha(t)\\
&+ \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q_{k_\alpha}^-|\big(|\sigma_\alpha| + |q_{k_\alpha}^-|\big) + |\sigma_\alpha|\sum_{i \neq k_\alpha}|q_i^-|\Big). \qquad (4.4.4.16)
\end{aligned}
$$

In Lemma 2.3.2, we showed that for a fixed rarefaction wave, the infimum of the discrepancy $D_\alpha(t)$ is $\sigma_\alpha^2/4$. From this we deduce that $\mathcal{O}(1)|\sigma_\alpha|^3 \leq \mathcal{O}(1)D_\alpha(t)$. Using this in (4.4.4.16) we obtain the final form of the estimate.

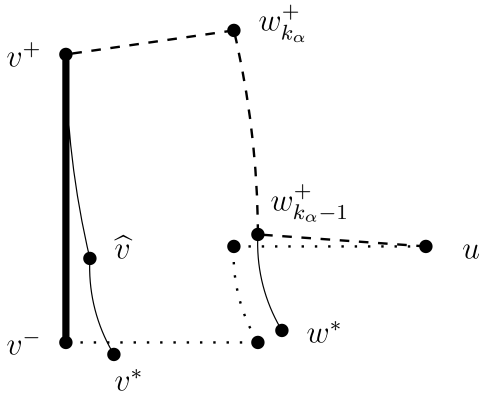

Figure 4.8: The virtual waves $q_i^\pm$, the rarefaction wave α, and the states in (4.4.5.1) for the seventh case.

4.4.5 Case 7: rarefaction with $q^-_{k_\alpha} < 0 < q^+_{k_\alpha}$

Assume that α is a pure rarefaction wave satisfying $q^-_{k_\alpha} < 0 < q^+_{k_\alpha}$. We demonstrate the estimate for terms in $|q_i^+|$.

Define the quantities

$$\begin{aligned}
\widetilde{w} &= S_{k_\alpha}(-\sigma_\alpha)(w^+_{k_\alpha}), & \widetilde{s} &= s_{k_\alpha}(w^+_{k_\alpha}, \widetilde{w}),\\
w^* &= S_{k_\alpha}(q^+_{k_\alpha} - \sigma_\alpha)(w^+_{k_\alpha-1}), & s^* &= s_{k_\alpha}(w^+_{k_\alpha-1}, w^*),\\
\widetilde{v} &= S_{k_\alpha}(-\sigma_\alpha)(v^+), & \widetilde{\eta} &= s_{k_\alpha}(v^+, \widetilde{v}),\\
\widehat{v} &= S_{k_\alpha}(-q^+_{k_\alpha})(v^+), & \widehat{\eta} &= s_{k_\alpha}(v^+, \widehat{v}),\\
v^* &= S_{k_\alpha}(q^+_{k_\alpha} - \sigma_\alpha)(\widehat{v}), & \eta^* &= s_{k_\alpha}(\widehat{v}, v^*),
\end{aligned} \tag{4.4.5.1}$$

and

$$\overline{v} = R_{k_\alpha}(-q^+_{k_\alpha})(v^+).$$

The states $\widetilde{v}$ and v^- are both a distance $-\sigma_\alpha$ from v^+. From the parameterization (1.1.15) we conclude that

$$\lambda_{k_\alpha}(\widetilde{v}) = \lambda_{k_\alpha}(v^-). \tag{4.4.5.2}$$

We now compare pairs of shock speeds defined by states along the shock curve separated by equal distances. We verify that

$$\begin{aligned}
|\widetilde{s} - \widetilde{\eta}| &= \left|s_{k_\alpha}(w^+_{k_\alpha}, \widetilde{w}) - s_{k_\alpha}(v^+, \widetilde{v})\right|\\
&= \mathcal{O}(1)|w^+_{k_\alpha} - v^+|\\
&= \mathcal{O}(1)\sum_{i \neq k_\alpha} |q_i^+|,
\end{aligned} \tag{4.4.5.3}$$

$$\begin{aligned}|s^+_{k_\alpha} - \widehat{\eta}| &= \big|s_{k_\alpha}(w^+_{k_\alpha-1}, w^+_{k_\alpha}) - s_{k_\alpha}(v^+, \widehat{v})\big| \\ &= \mathcal{O}(1)|w^+_{k_\alpha} - v^+| \\ &= \mathcal{O}(1)\sum_{i\neq k_\alpha}|q^+_i|, \end{aligned} \tag{4.4.5.4}$$

and

$$\begin{aligned}|s^* - \eta^*| &= |s_{k_\alpha}(w^+_{k_\alpha-1}, w^*) - s_{k_\alpha}(\widehat{v}, v^*)| \\ &= \mathcal{O}(1)|w^+_{k_\alpha-1} - \widehat{v}| \\ &= \mathcal{O}(1)|w^+_{k_\alpha} - v^+| \\ &= \mathcal{O}(1)\sum_{i\neq k_\alpha}|q^+_i|. \end{aligned} \tag{4.4.5.5}$$

In the next estimate, the pairs of states defining the shock speeds $s^-_{k_\alpha}$ and s^* are separated along S_{k_α} by the distances $q^-_{k_\alpha}$ and $q^+_{k_\alpha} - \sigma_\alpha$. With estimate (4.2.8) of Lemma 4.2.2, we find

$$\begin{aligned}|s^-_{k_\alpha} - s^*| =& \big|s_{k_\alpha}(w^-_{k_\alpha-1}, w^-_{k_\alpha}) - s_{k_\alpha}(w^+_{k_\alpha-1}, w^*)\big| \\ =& \mathcal{O}(1)\Big(|w^+_{k_\alpha-1} - w^-_{k_\alpha-1}| + |q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha|\Big) \\ =& \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| - |q^+_{k_\alpha}|\big) + \sum_{i\neq k_\alpha}|\sigma_\alpha||q^+_i|\Big). \end{aligned} \tag{4.4.5.6}$$

Since the shock speeds are, to second-order, the average of the characteristic speeds of the waves at the end states, we have the following three estimates

$$\Big|\widetilde{\eta} - \frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(\widetilde{v})\big)\Big| = \mathcal{O}(1)|\sigma_\alpha|^2, \tag{4.4.5.7}$$

$$\Big|\eta^* - \frac{1}{2}\big(\lambda_{k_\alpha}(v^*) + \lambda_{k_\alpha}(\widehat{v})\big)\Big| = \mathcal{O}(1)|q^+_{k_\alpha} - \sigma_\alpha|^2, \tag{4.4.5.8}$$

$$\Big|\widehat{\eta} - \frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(\widehat{v})\big)\Big| = \mathcal{O}(1)|q^+_{k_\alpha}|^2. \tag{4.4.5.9}$$

From the second-order tangency of shock and rarefaction curves, we deduce

$$\begin{aligned}|\widehat{v} - \overline{v}| =& \big|S_{k_\alpha}(-q^+_{k_\alpha})(v^+) - R_{k_\alpha}(-q^+_{k_\alpha})(v^+)\big| \\ =& \mathcal{O}(1)|q^+_{k_\alpha}|^3. \end{aligned} \tag{4.4.5.10}$$

Estimate (1.1.27) in the definition of the constant d, with $q = q^+_{k_\alpha}$, $p = \sigma_\alpha - q^+_{k_\alpha}$, and $u = w^+_{k_\alpha}$ states

$$d|\sigma_\alpha - q^+_{k_\alpha}| = d\big(|\sigma_\alpha| - |q^+_{k_\alpha}|\big) < s^+_{k_\alpha} - \widetilde{s}. \tag{4.4.5.11}$$

From the definition of W_{k_α} and the estimate $W_{k_\alpha} \le 2$, we have

$$\kappa_1\big(V(u)+V(v)\big) \le W_{k_\alpha} - 1 \le 1. \tag{4.4.5.12}$$

Using identity (2.2.9) and these inequalities, we find

$$\begin{aligned}\kappa_1\big(B^-_{k_\alpha}+B^+_{k_\alpha}\big) &\le \kappa_1\big(|\sigma_\alpha|+V(u)+V(v)\big)\\ &\le 2\kappa_1\big(V(u)+V(v)\big) \le 2.\end{aligned} \tag{4.4.5.13}$$

We now consider the expression $E_{k_\alpha}(\alpha)$ which we expand using the identity $A^-_{k_\alpha} = A^+_{k_\alpha}$:

$$\begin{aligned}E_{k_\alpha}(\alpha) =&\Big(1+\kappa_1\big(A^+_{k_\alpha}+B^+_{k_\alpha}\big)+\kappa_2\big(Q(u)+Q(v)\big)\Big)|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha)\\ &-\Big(1+\kappa_1\big(A^-_{k_\alpha}+B^-_{k_\alpha}\big)+\kappa_2\big(Q(u)+Q(v)\big)\Big)|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\\ =&\Big(1+\kappa_1A^+_{k_\alpha}+\kappa_2\big(Q(u)+Q(v)\big)\Big)\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha)-|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\ &+\kappa_1\big(B^-_{k_\alpha}+B^+_{k_\alpha}\big)|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha)\\ &-\kappa_1B^-_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha)+|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}.\end{aligned} \tag{4.4.5.14}$$

Beginning with the second term in (4.4.5.14), use identity (4.4.5.2) to write

$$\begin{aligned}&\kappa_1|q^+_{k_\alpha}|\big(B^-_{k_\alpha}+B^+_{k_\alpha}\big)(s^+_{k_\alpha}-\dot{x}_\alpha)\\ &\quad=\kappa_1|q^+_{k_\alpha}|\big(B^-_{k_\alpha}+B^+_{k_\alpha}+|\sigma_\alpha|\big)(s^+_{k_\alpha}-\widetilde{s})-\kappa_1|q^+_{k_\alpha}||\sigma_\alpha|(s^+_{k_\alpha}-\widetilde{s})\\ &\qquad+\kappa_1|q^+_{k_\alpha}|\big(B^-_{k_\alpha}+B^+_{k_\alpha}\big)\Big[\big(\widetilde{s}-\widetilde{\eta}\big)+\Big(\widetilde{\eta}-\frac{1}{2}\big(\lambda_{k_\alpha}(v^+)+\lambda_{k_\alpha}(v^-)\big)\Big)\\ &\qquad\qquad+\Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^+)+\lambda_{k_\alpha}(v^-)\big)-\dot{x}_\alpha\Big)\Big].\end{aligned} \tag{4.4.5.15}$$

Our parameterization (1.1.15) along shock curves implies that

$$0 \le s^+_{k_\alpha} - \widetilde{s} \le \lambda_{k_\alpha}(w^+_{k_\alpha}) - \lambda_{k_\alpha}(\widetilde{w}) = |\sigma_\alpha|. \tag{4.4.5.16}$$

In the following order, use the bounds (4.4.5.13), $|q^+_{k_\alpha}| \le |\sigma_\alpha|$, (4.4.5.16), and the bound on the discrepancy (2.3.16) to estimate the first term in the last equality of (4.4.5.15) as

$$\begin{aligned}\kappa_1|q^+_{k_\alpha}|\big(B^-_{k_\alpha}+B^+_{k_\alpha}+|\sigma_\alpha|\big)(s^+_{k_\alpha}-\widetilde{s}) \le&\, 3|q^+_{k_\alpha}||s^+_{k_\alpha}-\widetilde{s}|\\ \le&\, 3|\sigma_\alpha|^2\\ \le&\, 6D_\alpha(t).\end{aligned} \tag{4.4.5.17}$$

Apply estimate (4.4.5.11) to the second term in (4.4.5.15) to find

$$-\kappa_1|q^+_{k_\alpha}||\sigma_\alpha|(s^+_{k_\alpha}-\widetilde{s}) < -\kappa_1 d|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha|-|q^+_{k_\alpha}|\big). \tag{4.4.5.18}$$

To estimate the last term in (4.4.5.15), distribute the multiplier across the three terms inside the parenthesis. Combine (4.4.5.3), (4.4.5.13), and $|q^+_{k_\alpha}| < |\sigma_\alpha|$ to bound the first of these terms. For the second term, use (4.4.5.7), (4.4.5.13), and $|q^+_{k_\alpha}| < |\sigma_\alpha|$. Together, these two terms are bounded by

$$\mathcal{O}(1)|\sigma_\alpha|\sum_{i\neq k_\alpha}|q^+_i| + \mathcal{O}(1)|\sigma_\alpha|^3. \tag{4.4.5.19}$$

Referring to our explicit description of the discrepancy (2.3.12), we see that the last term in (4.4.5.15) is bounded above by $2D_\alpha(t)$. With estimates (4.4.5.17), (4.4.5.18), and (4.4.5.19) we have shown that

$$\begin{aligned}\kappa_1|q^+_{k_\alpha}|\big(B^-_{k_\alpha}+B^+_{k_\alpha}\big)(s^+_{k_\alpha}-\dot{x}_\alpha) \leq &-\kappa_1 d|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha|-|q^+_{k_\alpha}|\big)+8D_\alpha(t)\\ &+\mathcal{O}(1)\Big(|\sigma_\alpha|^3+|\sigma_\alpha|\sum_{i\neq k_\alpha}|q^+_i|\Big).\end{aligned} \tag{4.4.5.20}$$

We expand the third term of (4.4.5.14) as

$$\begin{aligned}&-\kappa_1 B^-_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha)+|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\ &\quad=-\kappa_1 B^-_{k_\alpha}\Big\{q^+_{k_\alpha}(s^+_{k_\alpha}-\dot{x}_\alpha)+(\sigma_\alpha-q^+_{k_\alpha})(s^-_{k_\alpha}-\dot{x}_\alpha)\\ &\qquad\qquad+(q^+_{k_\alpha}-q^-_{k_\alpha}-\sigma_\alpha)(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\ &\quad=-\kappa_1 B^-_{k_\alpha}\Big\{q^+_{k_\alpha}\big((s^+_{k_\alpha}-\widetilde{s})+(\widetilde{s}-\dot{x}_\alpha)\big)\\ &\qquad\qquad\quad+(\sigma_\alpha-q^+_{k_\alpha})\big((s^-_{k_\alpha}-s^*)+(s^*-\widetilde{s})+(\widetilde{s}-\dot{x}_\alpha)\big)\\ &\qquad\qquad\quad+(q^+_{k_\alpha}-q^-_{k_\alpha}-\sigma_\alpha)(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\ &\quad=-\kappa_1 B^-_{k_\alpha}\Big\{(\sigma_\alpha-q^+_{k_\alpha})(s^*-\widetilde{s})+q^+_{k_\alpha}(s^+_{k_\alpha}-\widetilde{s})\Big\}\\ &\qquad-\kappa_1 B^-_{k_\alpha}(\sigma_\alpha-q^+_{k_\alpha})(s^-_{k_\alpha}-s^*)-\kappa_1 B^-_{k_\alpha}\sigma_\alpha(\widetilde{s}-\dot{x}_\alpha)\\ &\qquad-\kappa_1 B^-_{k_\alpha}(q^+_{k_\alpha}-q^-_{k_\alpha}-\sigma_\alpha)(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}.\end{aligned} \tag{4.4.5.21}$$

From (4.4.5.12), $\kappa_1 B^-_{k_\alpha} \leq 1$, and we may ignore the coefficient in front of (4.4.5.21). The first and second terms in (4.4.5.21) give precisely the expression (4.2.19) of Lemma 4.2.5 with $\overline{w}=w^+_{k_\alpha-1}$, $\sigma=q^+_{k_\alpha}$, and $\sigma'=-\sigma_\alpha$:

$$\big|(q^+_{k_\alpha}-\sigma_\alpha)(s^*-\widetilde{s})-q^+_{k_\alpha}(s^+_{k_\alpha}-\widetilde{s})\big|=\mathcal{O}(1)|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha|-|q^+_{k_\alpha}|\big). \tag{4.4.5.22}$$

The third term in the last equality of (4.4.5.21) is handled by appealing to estimate (4.4.5.6). In the last term, $(s_{k_\alpha}^- - \dot{x}_\alpha)$ is bounded since $|\dot{x}_\alpha|$ is bounded by Λ. Using this and equation (4.2.8), we bound the last term. Rewrite the fourth term as

$$\begin{aligned}\sigma_\alpha(\widetilde{s} - \dot{x}_\alpha) = \sigma_\alpha \Big[\big(\widetilde{s} - \widetilde{\eta}\big) + \Big(\widetilde{\eta} - \frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(\widetilde{v})\big)\Big) \\ + \Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(\widetilde{v})\big) - \dot{x}_\alpha\Big)\Big]. \end{aligned} \tag{4.4.5.23}$$

Use estimates (4.4.5.3), (4.4.5.7), and inequality (2.3.15) of Lemma 2.3.2 to demonstrate

$$\sigma_\alpha(\widetilde{s} - \dot{x}_\alpha) \le \mathcal{O}(1)|\sigma_\alpha| \sum_{i \ne k_\alpha} |q_i^+| + \mathcal{O}(1)|\sigma_\alpha|^3 + D_\alpha(t). \tag{4.4.5.24}$$

For the third term in (4.4.5.14), these observations show that

$$\begin{aligned} -\kappa_1 B_{k_\alpha}^- \Big\{ |q_{k_\alpha}^+|(s_{k_\alpha}^+ - \dot{x}_\alpha) + |q_{k_\alpha}^-|(s_{k_\alpha}^- - \dot{x}_\alpha) \Big\} \\ \le D_\alpha(t) + \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q_{k_\alpha}^+|\big(|\sigma_\alpha| - |q_{k_\alpha}^+|\big) + \sum_{i \ne k_\alpha} |\sigma_\alpha||q_i^+| \Big). \end{aligned} \tag{4.4.5.25}$$

The coefficient of the first term in (4.4.5.14) satisfies

$$1 + \kappa_1 A_{k_\alpha}^+ + \kappa_2\big(Q(u) + Q(v)\big) \le 2. \tag{4.4.5.26}$$

For convenience, we expand the first term in (4.4.5.14) without this coefficient. Using the fact that $q_{k_\alpha}^-$, $q_{k_\alpha}^+$, and σ_α are positive, we write

$$\begin{aligned} |q_{k_\alpha}^+|(s_{k_\alpha}^+ - \dot{x}_\alpha) - |q_{k_\alpha}^-|(s_{k_\alpha}^- - \dot{x}_\alpha) \\ = q_{k_\alpha}^+ \Big[\big(s_{k_\alpha}^+ - \widehat{\eta}\big) + \Big(\widehat{\eta} - \frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(\widehat{v})\big)\Big) \\ + \Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(\widehat{v})\big) - \dot{x}_\alpha\Big)\Big] \\ - (\sigma_\alpha - q_{k_\alpha}^+)\Big[\big(s_{k_\alpha}^- - s^*\big) + \big(s^* - \eta^*\big) \\ + \Big(\eta^* - \frac{1}{2}\big(\lambda_{k_\alpha}(\widehat{v}) + \lambda_{k_\alpha}(v^*)\big)\Big) \\ + \Big(\frac{1}{2}\big(\lambda_{k_\alpha}(\widehat{v}) + \lambda_{k_\alpha}(v^*)\big) - \dot{x}_\alpha\Big)\Big] \\ - (q_{k_\alpha}^+ - q_{k_\alpha}^- - \sigma_\alpha)(s_{k_\alpha}^- - \dot{x}_\alpha). \end{aligned} \tag{4.4.5.27}$$

In our count of the terms in this expression, we will assume that distributivity has been applied across the square parenthesis. For the first term, combine $|q^+_{k_\alpha}| \le |\sigma_\alpha|$ with (4.4.5.4) to obtain the bound $\mathcal{O}(1)|\sigma_\alpha|\sum|q^+_i|$. Estimate (4.4.5.9) and $|q^+_{k_\alpha}| \le |\sigma_\alpha|$ imply that the second term is of order $\mathcal{O}(1)|\sigma_\alpha|^3$. Also note that $|\sigma_\alpha - q^+_{k_\alpha}| \le |\sigma_\alpha|$. Combining this bound with (4.4.5.5), (4.4.5.6), and (4.4.5.8) we find that the fourth, fifth and sixth terms are of order

$$\mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| - |q^+_{k_\alpha}|\big) + \sum_{i\neq k_\alpha}|\sigma_\alpha||q^+_i|\Big). \tag{4.4.5.28}$$

Equation (4.2.8) of Lemma 4.2.2 shows that the last term is also of this order. As for the terms involving $\dot{x}_\alpha$, notice that our parameterization along the shock curves provides us with the identities $\lambda_{k_\alpha}(\widehat{v}) = \lambda_{k_\alpha}(\overline{v})$ and $\lambda_{k_\alpha}(v^*) = \lambda_{k_\alpha}(v^-)$. Summarizing, we have shown that the first term in (4.4.5.14) satisfies

$$\begin{aligned}\Big(1 + \kappa_1 A^+_{k_\alpha} + \kappa_2\big(Q(u) + Q(v)\big)\Big)&\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\}\\ \le 2&\left|q^+_{k_\alpha}\Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(\overline{v})\big) - \dot{x}_\alpha\Big)\right.\\ &\left.- (\sigma_\alpha - q^+_{k_\alpha})\Big(\frac{1}{2}\big(\lambda_{k_\alpha}(\overline{v}) + \lambda_{k_\alpha}(v^-)\big) - \dot{x}_\alpha\Big)\right|\\ &+ \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| - |q^+_{k_\alpha}|\big) + \sum_{i\neq k_\alpha}|\sigma_\alpha||q^+_i|\Big).\end{aligned} \tag{4.4.5.29}$$

The first term above is twice the quadratic rarefaction remainder $Q_{\text{rar}}(q^+_{k_\alpha}, \dot{x}_\alpha)$ and by Lemma 4.2.7 is bounded by the discrepancy $2D_\alpha(t)$. The final estimate of $E_{k_\alpha}(\alpha)$ combines the estimates (4.4.5.20), (4.4.5.25), (4.4.5.29) for the three terms in (4.4.5.14) with $\mathcal{O}(1)|\sigma_\alpha|^3 = \mathcal{O}(1)D_\alpha(t)$. The conclusion is

$$\begin{aligned}E_{k_\alpha}(\alpha) \le &- \kappa_1 d|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| - |q^+_{k_\alpha}|\big) + \mathcal{O}(1)D_\alpha(t)\\ &+ \mathcal{O}(1)\Big(|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| - |q^+_{k_\alpha}|\big) + |\sigma_\alpha|\sum_{i\neq k_\alpha}|q^+_i|\Big).\end{aligned} \tag{4.4.5.30}$$

The proof of Lemma 4.4.1 is complete and contained in Subsections 4.4.1-4.4.5.
■

4.5 Isolated Waves: Proof of Lemma 4.1.1

We now present the proof of Lemma 4.1.1 under the assumption that $\alpha \in \mathcal{W}\big(v(t)\big)$ is an isolated wave.

Proof We must first demonstrate that Φ is well-defined under the conditions of this lemma.

Choose a value for κ_1, say $\kappa_1 = 1$. Let $\delta_0 = \delta_0(\kappa_1)$ be the positive constant of Lemma 2.2.2 and let δ_1, δ_2 be the constants of Lemmas 4.2.4 and 4.2.6. Define

$\delta_0' = 1/2 \min\{\delta_0, \delta_1, \delta_2\}$ and suppose that the front-tracking approximations u and v satisfy

$$||u(0)||_{\mathrm{TV}}, ||v(0)||_{\mathrm{TV}} < \delta_0'. \tag{4.5.1}$$

Notice that because both $u(0)$ and $v(0)$ are assumed to be in $L^1(\mathbb{R}, \Omega)$,

$$u(0) - v(0) \in L^1(\mathbb{R}, \Omega). \tag{4.5.2}$$

Moreover, both $u(0)$ and $v(0)$ must vanish at infinity and therefore the total variation bound also implies an L^∞ bound

$$|| \cdot ||_{L^\infty} \leq || \cdot ||_{\mathrm{TV}}. \tag{4.5.3}$$

Using (4.5.1) and (4.5.3), we have

$$||u(0)||_{L^\infty} + ||u(0)||_{\mathrm{TV}} \leq 2||u(0)||_{\mathrm{TV}} < 2\delta_0' \leq \delta_0. \tag{4.5.4}$$

Similarly, $||u(0)||_{L^\infty} + ||u(0)||_{\mathrm{TV}} < \delta_0$.

Conditions (4.5.1), therefore imply the initial bounds (4.5.4) for both u and v. These are precisely the conditions of Lemma 2.2.2 ensuring that $\Phi\big(u(t), v(t)\big)$ is well-defined. In fact, the approximations u and v also satisfy the conditions on the total variation of Lemmas 4.2.4 and 4.2.6 and therefore satisfy the assumption (4.3.1) stated at the beginning of Section 4.3. The estimates of Lemmas 4.3.1 and 4.4.1 hold for u and v.

Suppose that α is shock wave satisfying $0 < q^+_{k_\alpha} < q^-_{k_\alpha}$. Summing equation (4.3.3) over $k \neq k_\alpha$ with (4.4.1), all with the positive sign s, we obtain

$$\begin{aligned}\sum_{k=1}^{n} E_k(\alpha) \leq & -\kappa_1 d|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big) - \kappa_1 d|\sigma_\alpha| \sum_{i\neq k_\alpha} |q^+_i| + \mathcal{O}(1)D_\alpha(t) \\ & + \mathcal{O}(1)\Big(|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big) + |\sigma_\alpha| \sum_{i\neq k_\alpha} |q^+_i|\Big).\end{aligned} \tag{4.5.5}$$

There exists κ_1, depending only on the system (1.1.2), such that

$$\mathcal{O}(1) \leq \kappa_1 d,$$

where $\mathcal{O}(1)$ is the coefficient of the last term in (4.5.5). For this value of κ_1 recompute $\delta_0 = \delta_0(\kappa_1)$ of Lemma 2.2.2 and δ_0'. Assuming that both u and v satisfy the bounds (4.5.1) for this new value of δ_0', we have

$$\sum_{k=1}^{n} E_k(\alpha) \leq \mathcal{O}(1)D_\alpha(t). \tag{4.5.6}$$

If α is a shock wave satisfying either $q^+_{k_\alpha} < q^-_{k_\alpha} < 0$ or $q^+_{k_\alpha} < 0 < q^-_{k_\alpha}$, sum the estimates of Lemmas 4.3.1 and 4.4.1 to obtain another lower bound for κ_1 of the form $\mathcal{O}(1) \leq \kappa_1 d$. If α is a rarefaction or linearly degenerate wave, Lemmas 4.3.1 and 4.4.1 also imply a lower bound on the size of κ_1.

In all these cases, the lower bound on κ_1 is computable entirely from the system (1.1.2). In particular, the constant $\delta_0' = \delta_0'(\kappa_1)$ also depends only on (1.1.2). This demonstrates the lemma. ■

4.6 Non-Isolated Waves

In general, discontinuities in front-tracking approximations are composed of more than one wave. The construction of Section 2.1 gives rise to non-isolated waves when condition (2.1.13) forces the introduction of an approximation which limits the number of outgoing discontinuities. To demonstrate Lemma 4.1.1 for such discontinuities we construct an intermediate approximation where the single discontinuity is replaced locally by n isolated waves traveling within a distance ϵ of each other. We then show that our earlier estimates pass to the limit as $\epsilon \to 0$. This construction is formalized below.

Consider a front-tracking approximation u with a single discontinuity traveling along the trajectory $z(t)$. Since we only need to demonstrate Lemma 4.1.1 for times when discontinuities are not colliding, we can assume that there is only one discontinuity in a neighborhood of $\big(t_0, z(t_0)\big) = (t_0, z_0)$. More specifically, we assume that there exists ϵ such that the box

$$B_\epsilon(t_0, z_0) = \big[t_0 - \epsilon, t_0 + \epsilon\big] \times \big[z_0 - \Lambda\epsilon, z_0 + \Lambda\epsilon\big] \tag{4.6.1}$$

contains only one discontinuity. Since Λ is the CFL number of the front-tracking algorithm, defined in (2.1.14), the discontinuity enters and exits the box along the lower and upper edges.

We denote the n waves in $\mathcal{W}\big(v(t)\big)$ at located along $z(t)$ by $\alpha(1), \alpha(2), \ldots, \alpha(n)$ and order them to satisfy $k_{\alpha(i)} < k_{\alpha(i+1)}$. Recall Definition 2.1.3 of the left and right hand states of a wave. It states that if v^- and v^+ are the left and right sided limits of v at $z(t)$, then

$$v^- = v^-_{\alpha(1)}, \qquad v^+ = v^+_{\alpha(n)}, \tag{4.6.2}$$

and

$$v^+_{\alpha(i)} = T_i(\sigma_{\alpha(i)})(v^-_{\alpha(i)}), \qquad v^+_{\alpha(i)} = v^-_{\alpha(i+1)}. \tag{4.6.3}$$

Note that the states $v^-_{\alpha(1)}, \ldots, v^-_{\alpha(n)}, v^+_{\alpha(n)}$ are the $n+1$ intermediate states in the solution of the Riemann problem between v^- and v^+.

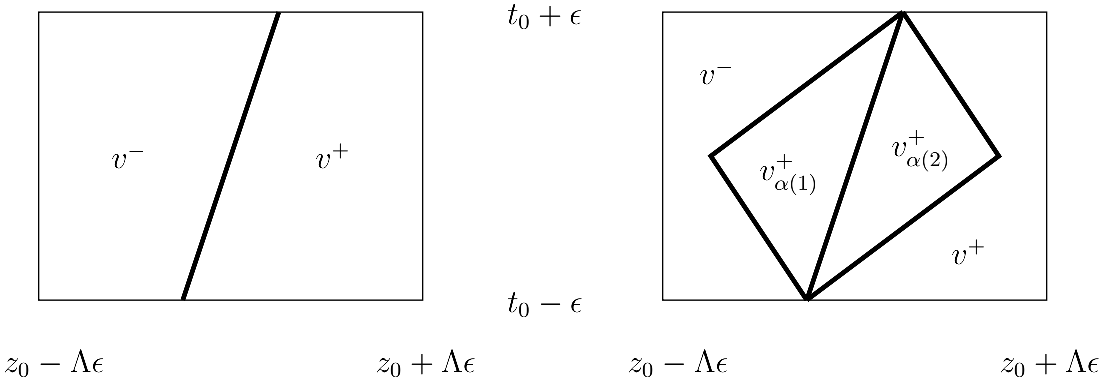

Figure 4.9: The bold line in the left box represents the trajectory z of the discontinuity in the approximation v. The right box represents the trajectories τ_i and the value of V_ϵ in each region bounded by these curves.

We replace the discontinuity $z(t)$ in $B_\epsilon(t_0, z_0)$ by n isolated waves traveling along the trajectories $\tau_1(t), \dots, \tau_n(t)$ given below. For $t \in \big[t_0-\epsilon, t_0+\epsilon\big]$, define

$$\tau_i(t) = z(t) + \epsilon\big(\epsilon - |t-t_0|\big)\cdot\left(\frac{2i-n-1}{2n-2}\right). \tag{4.6.4}$$

These curves are disjoint, $\tau_i(t) < \tau_{i+1}(t)$, except at the boundary of $B_\epsilon(t_0, z_0)$. In fact, they are separated by precisely

$$\big|\tau_i(t) - \tau_{i+1}(t)\big| = \frac{\epsilon}{n-1}\big(\epsilon - |t-t_0|\big).$$

In particular, the separation is of order ϵ^2 and for ϵ sufficiently small the trajectories τ_i are contained inside the box.

Definition 4.6.1 *The ϵ-splitting of v at (t_0, z_0) is the piecewise constant function*

$$V_\epsilon(t,x) = \begin{cases} v(t,x), & \textit{if } (t,x) \notin B_\epsilon(t_0,z_0), \\ v^+_{\alpha(i)} = v^-_{\alpha(i+1)}, & \textit{if } (t,x) \textit{ is between the curves } \tau_i \textit{ and } \tau_{i+1}, \\ v^-, & \textit{if } (t,x) \textit{ is to the left of } \tau_1, \\ v^+, & \textit{if } (t,x) \textit{ is to the right of } \tau_n. \end{cases} \tag{4.6.5}$$

The discontinuities in v and V_ϵ inside the domain $B_\epsilon(t_0, z_0)$ are illustrated in Figure 4.9.

The ϵ-splitting V_ϵ contains a single discontinuity at $z(t_0-\epsilon)$ which separates inside $B_\epsilon(t_0, z_0)$ into n isolated waves. This process is consistent with the first approximate Riemann solver used in the front-tracking algorithm. When the waves merge back together at $z(t_0+\epsilon)$, one is essentially appealing to the second approximate Riemann solver previously used under condition (2.1.13). The ϵ-splitting is therefore also a front-tracking approximation and the results of Lemmas 2.1.1 and 2.2.2 also hold. In particular $\Phi(u, V_\epsilon)$ is well-defined for the same total variation bounds as the admissible pair u and v.

Before proceeding, we introduce some notation to help distinguish between quantities appearing in the construction of the functionals $\Phi(u,v)$ and $\Phi(u,V_\epsilon)$. Slightly abusing our notation, let $\alpha(i)$ denote both the wave of the i-th family at $z(t)$ in $\mathcal{W}\big(v(t)\big)$ and the isolated wave in $\mathcal{W}\big(V_\epsilon(t)\big)$ traveling along the trajectory τ_i. Quantities used in the definition of the functional $\Phi(u,V_\epsilon)$ will be written with a further subscript ϵ. In particular, at any time t

$$\Phi\big(u(t),V_\epsilon(t)\big)=\sum_{k=1}^{n}\int_{-\infty}^{\infty}|q_{k,\epsilon}(x)|W_{k,\epsilon}(x)\;dx.$$

Recall the notation,

$$q_k^{\beta\pm}=\lim_{x\to x_\beta\pm}q_k(x),$$

which will also be used for any function of x.

Lemma 4.6.1

$$\lim_{\epsilon\to0}\sum_{i=1}^{n}\sum_{k=1}^{n}E_{k,\epsilon}\big(\alpha(i)\big)=\sum_{k=1}^{n}E_k(z_0). \tag{4.6.6}$$

Proof For $\alpha(i)$ and $\alpha(i+1)$, compare the two contributions from the time derivative of $\Phi\big(u(t),V_\epsilon(t)\big)$:

$$\begin{aligned}E_{k,\epsilon}\big(\alpha(i)\big)=&|q_{k,\epsilon}^{\alpha(i)+}|W_{k,\epsilon}^{\alpha(i)+}\big(s_{k,\epsilon}^{\alpha(i)+}-\dot\tau_i\big)\\&-|q_{k,\epsilon}^{\alpha(i)-}|W_{k,\epsilon}^{\alpha(i)-}\big(s_{k,\epsilon}^{\alpha(i)-}-\dot\tau_i\big),\end{aligned}\tag{4.6.7}$$

$$\begin{aligned}E_{k,\epsilon}\big(\alpha(i+1)\big)=&|q_{k,\epsilon}^{\alpha(i+1)+}|W_{k,\epsilon}^{\alpha(i+1)+}\big(s_{k,\epsilon}^{\alpha(i+1)+}-\dot\tau_{i+1}\big)\\&-|q_{k,\epsilon}^{\alpha(i+1)-}|W_{k,\epsilon}^{\alpha(i+1)-}\big(s_{k,\epsilon}^{\alpha(i+1)-}-\dot\tau_{i+1}\big).\end{aligned}\tag{4.6.8}$$

Using (4.6.3),

$$q_{k,\epsilon}^{\alpha(i)+}=q_k(u,v_{\alpha(i)}^{+})=q_k(u,v_{\alpha(i+1)}^{-})=q_{k,\epsilon}^{\alpha(i+1)-}. \tag{4.6.9}$$

The same procedure also shows that

$$s_{k,\epsilon}^{\alpha(i)+}=s_{k,\epsilon}^{\alpha(i+1)-},\ \text{and}\ W_{k,\epsilon}^{\alpha(i)+}=W_{k,\epsilon}^{\alpha(i+1)-}. \tag{4.6.10}$$

Moreover, notice that as $\epsilon \to 0$ both $\dot{\tau}_i$ and $\dot{\tau}_{i+1}$ approach $\dot{z}$. For fixed k, in the limit as $\epsilon \to 0$, the terms on the left hand side of (4.6.6) form a telescoping sum and only the first and last terms remain. The limit can therefore be reduced to

$$\begin{aligned}\lim_{\epsilon\to 0}\sum_{i=1}^{n}\sum_{k=1}^{n} E_{k,\epsilon}\big(\alpha(i)\big) =&\lim_{\epsilon\to 0}\sum_{k=1}^{n} \big|q_{k,\epsilon}^{\alpha(n)+}\big|W_{k,\epsilon}^{\alpha(n)+}\big(s_{k,\epsilon}^{\alpha(n)+}-\dot{z}\big)\\ &\qquad - \big|q_{k,\epsilon}^{\alpha(1)-}\big|W_{k,\epsilon}^{\alpha(1)-}\big(s_{k,\epsilon}^{\alpha(1)-}-\dot{z}\big). \end{aligned} \tag{4.6.11}$$

From the definition of the ϵ-splitting

$$\begin{aligned} q_{k,\epsilon}^{\alpha(1)-} &= q_k(u, v_{\alpha(1)}^{-}) = q_k(u, v^-) = q_k^{z-}, \\ q_{k,\epsilon}^{\alpha(n)+} &= q_k(u, v_{\alpha(n)}^{+}) = q_k(u, v^+) = q_k^{z+}. \end{aligned} \tag{4.6.12}$$

Similarly,

$$s_{k,\epsilon}^{\alpha(1)-} = s_k^{z-}, \text{ and } s_{k,\epsilon}^{\alpha(n)+} = s_k^{z+}. \tag{4.6.13}$$

We now examine the weight

$$W_{k,\epsilon}^{\alpha(1)-} = 1 + \kappa_1\big(A_{k,\epsilon}^{\alpha(1)-} + B_{k,\epsilon}^{\alpha(1)-}\big) + \kappa_2\big(Q(u) + Q(V_\epsilon)\big). \tag{4.6.14}$$

We know that $Q(V_\epsilon) = Q(v)$ since both approximations contain the same waves. From the definition of A_i, equation (2.2.4), we find

$$\begin{aligned}\lim_{\epsilon\to 0} A_{k,\epsilon}^{\alpha(1)-} =&\lim_{\epsilon\to 0}\Bigg(\sum_{\substack{\beta\in\mathcal{W}(u)\bigcup\mathcal{W}(V_\epsilon)\\ x_\beta<\tau_1,\, k_\beta>k}} + \sum_{\substack{\beta\in\mathcal{W}(u)\bigcup\mathcal{W}(V_\epsilon)\\ x_\beta\geq\tau_1,\, k_\beta<k}}\Bigg)|\sigma_\beta| \\ =&A_k^{z-}. \end{aligned} \tag{4.6.15}$$

One can also show that

$$\lim_{\epsilon\to 0} A_{k,\epsilon}^{\alpha(n)+} = A_k^{z+}. \tag{4.6.16}$$

If k is a genuinely nonlinear family, then

$$\begin{aligned}\lim_{\epsilon\to 0} B_{k,\epsilon}^{\alpha(1)-} =&\lim_{\epsilon\to 0}\begin{cases}\Big(\sum_{\substack{\beta\in\mathcal{W}(u)\\ x_\beta<\tau_1,\, k_\alpha=k}} + \sum_{\substack{\beta\in\mathcal{W}(V_\epsilon)\\ x_\beta\geq\tau_1,\, k_\alpha=k}}\Big)|\sigma_\beta| & \text{if } q_{k,\epsilon}^{\alpha(1)-}<0,\\ \Big(\sum_{\substack{\beta\in\mathcal{W}(V_\epsilon)\\ x_\beta<\tau_1,\, k_\beta=k}} + \sum_{\substack{\beta\in\mathcal{W}(u)\\ x_\beta\geq\tau_1,\, k_\beta=k}}\Big)|\sigma_\beta| & \text{if } q_{k,\epsilon}^{\alpha(1)-}>0,\end{cases} \\ =&B_k^{z-}. \end{aligned} \tag{4.6.17}$$

In the same manner, one verifies that

$$\lim_{\epsilon\to 0} B_{k,\epsilon}^{\alpha(n)+} = B_k^{z+}. \tag{4.6.18}$$

Applying identities (4.6.12)-(4.6.18) to equation (4.6.11) demonstrates the lemma. ■

Lemma 4.6.2

$$\lim_{\epsilon\to 0} D_{\alpha(i),\epsilon}(t_0) = D_{\alpha(i)}(t_0). \tag{4.6.19}$$

Proof If $\alpha(i)$ is shock wave or a linearly degenerate wave, then use formula (2.3.11) to compute

$$\begin{aligned}\lim_{\epsilon\to 0} D_{\alpha(i),\epsilon}(t_0) =& \lim_{\epsilon\to 0} |\sigma_{\alpha(i)}|\left|\dot{\tau}_i - s_i(v_{\alpha(i)}^-, v_{\alpha(i)}^+)\right| \\ =& |\sigma_{\alpha(i)}|\left|\dot{z} - s_i(v_{\alpha(i)}^-, v_{\alpha(i)}^+)\right| \\ =& D_{\alpha(i)}(t_0).\end{aligned} \tag{4.6.20}$$

If α is a rarefaction wave, repeat the calculation with formula (2.3.12). ■

Proof of Lemma 4.1.1 for non-isolated waves. It suffices to demonstrate this lemma at a point (t_0, z_0). Under the same conditions as those in the proof for isolated waves, we take two front-tracking approximations u and v and construct the ϵ-splitting V_ϵ for the discontinuity at (t_0, z_0). The lemma for isolated waves states that

$$\sum_{i=1}^{n}\sum_{k=1}^{n} E_{k,\epsilon}\big(\alpha(i)\big) \leq \mathcal{O}(1)\sum_{i=1}^{n} D_{\alpha(i),\epsilon}(t_0). \tag{4.6.21}$$

Lemmas 4.6.1 and 4.6.2 show that in the limit as ϵ goes to zero,

$$\sum_{k=1}^{n} E_k(z_0) \leq \mathcal{O}(1)\sum_{i=1}^{n} D_{\alpha(i)}(t_0). \tag{4.6.22}$$

■

Chapter 5

Conclusion

A posteriori error estimation for hyperbolic equations is a growing field and many new problems still need to be tackled. Our a posteriori error estimate for approximate solutions to hyperbolic systems of nonlinear conservation laws is the first of its kind and hopefully will help to stimulate the construction of estimates for more commonly used approximations. In particular, our method separates the numerical error, measured by the discrepancy, from the production of entropy, controlled by terms of order κ_1, and which are inherent to the physical system. Our proof has also provided several arguments supporting the use of discrepancies, namely its relationship to the residual in section 2.3, its equivalence to the error estimator of Cockburn in Theorem 3.1.1, and most importantly its explicit appearance in Lemmas 4.2.6 and 4.2.7.

Our work suggests that the recent stability theories for hyperbolic systems of conservation laws [12, 23, 34] could have important applications to a posteriori error estimation. In this direction, we mention the possibility of developing a stability theory for the Godunov scheme [7]. Given this schemes' central role in the numerics of conservation laws, such a technique would surely have an immediate impact on error estimation for most popular schemes. In [40], Di Perna constructed a stability theory for approximations combining finite differencing in smooth regions and Glimm's scheme for the propagation of shocks. With the help of the ideas of Liu and Yang, it might be possible to update the theory and produce results for the front-tracking method of Richtmeyer and von Neumann. In the long term, one could attempt to study other types of hyperbolic equations or to explore stability via the method of compensated compactness. Many researchers are actively working on these problems and the results could lead to significant improvement in the simulation of physical processes modeled by hyperbolic conservation laws.

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